Examples of positive operators in Krein space with 0 a regular critical point of infinite rank

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It is shown that the operators associated with the perturbed wave equation in $\mathbb{R}^n$ and with the elliptic operators with an indefinite weight function and mildly varying coefficients on $\mathbb{R}^n$ are similar to a selfadjoint operator in a Hilbert space. These operators have the whole $\mathbb{R}$ as the spectrum. It is shown that they are positive operators in corresponding Krein spaces, and the whole problem is reduced to showing that 0 is not a singular critical point.

1. Introduction

Let $\mathcal{K}$ be a Krein space, A a positive operator in $\mathcal{K}$ with nonempty resolvent set. Then A has a spectral function with the only possible critical points being 0 and ∞. In [3] we found sufficient conditions for a perturbation B in order that $A_1 = A + B$ be also a positive operator with nonempty resolvent set and that the nonsingularity of 0 and/or ∞ persists under this perturbation. We refer to [8] for the definitions and properties of Krein space operators.

In this note we give examples of an operator A and a perturbation B such that both 0 and ∞ are regular critical points of $A_1 = A+B$ and hence A_1 is similar to a selfadjoint operator in a Hilbert space. Note that in these examples both 0 and ∞ are critical points of infinite rank, i.e. there does not exist a neighbourhood Δ of one of these two points such that $E(\Delta)\mathcal{K}$ is a Pontryagin space. The examples are the operator associated with the perturbed wave equation and an elliptic operator with an indefinite weight. The wave equation example implies a well-posedness result which seems to be difficult to prove without the Krein space theory. For other examples of 0 being a regular critical point of a positive operator we refer to [5].

In [3] we have proved the following result:

Theorem 1.1. *Let $(\mathcal{K}, [\,\cdot\,|\,\cdot\,])$ be a Krein space and a and b two symmetric forms in $\mathcal{K}$. Assume that a is closed, symmetric and positive (by positive we mean $a(x) > 0$ for all $x \in \mathcal{D}(a)$, $x \neq 0$). Further assume that $\mathcal{D}(a) \subseteq \mathcal{D}(b)$ and that there exist real numbers α and β such that*

$$\alpha \leq \frac{b(x)}{a(x)} \leq \beta \quad \text{for all} \quad x \in \mathcal{D}(a)\ . \tag{1.1}$$

Let

$$a_\kappa = a + \kappa b, \ \ \kappa \in \mathbb{R}.$$

For $\kappa\alpha > -1$ the form a_κ is also a closed positive symmetric form on $\mathcal{D}(a)$. Let A and A_κ be the positive selfadjoint operators associated in $(\mathcal{K}, [\,\cdot\,|\,\cdot\,])$ with a and a_κ, resp. (see [7]). Assume that the operator A has nonempty resolvent set and that ∞ is not a singular critical point of A.

There exist real numbers $\kappa^\pm$ such that $\kappa^- < 0 < \kappa^+$ and that for $\kappa_- < \kappa < \kappa_+$ the operator A_κ has nonempty resolvent set and that ∞ is not its singular critical point. Moreover following statements are equivalent.

(i) *0 is not a singular critical point of A.*

(ii) *0 is not a singular critical point of A_κ.*

(iii) *A is similar to a selfadjoint operator in $(\mathcal{K}, (\,\cdot\,|\,\cdot\,))$.*

(iv) *A_κ is similar to a selfadjoint operator in $(\mathcal{K}, (\,\cdot\,|\,\cdot\,))$.*

2. Perturbed wave equation

The example to be described is an extension of the example in [2, 6].

Let $\mathcal{G}$ be a Hilbert space with a scalar product $(\,\cdot\,|\,\cdot\,)$, H a nonnegative injective selfadjoint operator in $\mathcal{G}$. For $\alpha \in \mathbb{R}$ let $\mathcal{G}_\alpha$ be the Hilbert space completion of $(\mathcal{D}(H^\alpha), (H^\alpha\,\cdot\,|H^\alpha\cdot))$. Denote by $\|\cdot\|_\alpha$ the norm of this Hilbert space. The operator H^β can be extended to an isometry between $\mathcal{G}_\alpha$ and $\mathcal{G}_{\alpha-\beta}$. Denote by $\mathcal{H}$ the Hilbert space $\mathcal{G}_{1/4} \oplus \mathcal{G}_{-1/4}$ and by $\langle\,\cdot\,|\,\cdot\,\rangle$ its natural scalar product. If $x \in \mathcal{G}_{1/4}$ then $|(x|y)| \leq \|x\|_{1/4}\|y\|_{-1/4}$ $(y \in \mathcal{G})$. Therefore the scalar product $(\,\cdot\,|\,\cdot\,)$ can be extended by continuity from $\mathcal{G}_{1/4} \times \mathcal{G}$ to $\mathcal{G}_{1/4} \times \mathcal{G}_{-1/4}$ and similarly from $\mathcal{G} \times \mathcal{G}_{1/4}$ to $\mathcal{G}_{-1/4} \times \mathcal{G}_{1/4}$. Define an indefinite scalar product on $\mathcal{H}$ by

$$[x|y] = (x_1|y_2) + (x_2|y_1), \quad x = (x_1, x_2), \ y = (y_1, y_2) \in \mathcal{H}.$$

The space $\mathcal{H}$ with the indefinite scalar product $[\,\cdot\,|\,\cdot\,]$ is a Krein space. The fundamental symmetry is

$$\mathbf{J} = \begin{bmatrix} 0 & H^{-1/2} \\ H^{1/2} & 0 \end{bmatrix}.$$

Define the operator $\mathbf{A}$ in $\mathcal{H}$ on $\mathcal{D}(\mathbf{A}) = \mathcal{G}_{3/4} \oplus \mathcal{G}_{1/4}$ by

$$\mathbf{A} = \begin{bmatrix} 0 & I \\ H & 0 \end{bmatrix}.$$

The operator $\mathbf{A}$ is a selfadjoint operator in $(\mathcal{H}, [\,\cdot\,|\,\cdot\,])$. Since

$$[\mathbf{A}x|x] = (Hx_1|x_1) + (x_2|x_2), \quad x = (x_1, x_2) \in \mathcal{D}(\mathbf{A}), \tag{2.1}$$

the operator $\mathbf{A}$ is positive in $(\mathcal{H}, [\cdot|\cdot])$. The form $[\mathbf{A}x|y]$, $x, y \in \mathcal{D}(\mathbf{A})$ is closable. Let $\mathbf{a}$ be its closure. It follows from (2.1) that the domain of $\mathbf{a}$ is $\mathcal{D}(\mathbf{a}) = \mathcal{G}_{1/2} \oplus \mathcal{G}$ and that

$$\mathbf{a}(x, y) = (H^{1/2}x_1|H^{1/2}y_1) + (x_2|y_2), \quad x = (x_1, x_2),\ y = (y_1, y_2) \in \mathcal{D}(\mathbf{a}).$$

Since the operators $\mathbf{A}$ and $\mathbf{J}$ commute we have:

Lemma 2.1. *The operator* $\mathbf{A}$ *is similar to a selfadjoint operator in* $\mathcal{H}$. *In particular, neither* ∞ *nor* 0 *is a singular critical point of* $\mathbf{A}$.

Let q and V be symmetric $H^{1/2}-$ bounded operators in $\mathcal{G}$. We define the form $\mathbf{b}$ on $\mathcal{D}(\mathbf{b}) = \mathcal{D}(\mathbf{a})$

$$\mathbf{b}(x, y) = (qx_1|qy_1) + (Vx_1|y_2) + (x_2|Vy_1),\ x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{D}(\mathbf{a})\ .$$

The operator formally associated with the form $\mathbf{b}$ in $(\mathcal{H}, [\cdot|\cdot])$ is

$$\mathbf{B} = \begin{bmatrix} V & 0 \\ q^2 & V \end{bmatrix}.$$

Lemma 2.2. *Under above assumptions*

$$\alpha \le \frac{\mathbf{b}(x)}{\mathbf{a}(x)} \le \beta \quad \text{for all} \quad x \in \mathcal{D}(\mathbf{a})\ . \tag{2.2}$$

where $\alpha = \frac{1}{2}(\|qH^{-1/2}\|^2 - (\|qH^{-1/2}\|^4 + 4\|VH^{-1/2}\|^2)^{1/2})$, $\beta = \frac{1}{2}(\|qH^{-1/2}\|^2 + (\|qH^{-1/2}\|^4 + 4\|VH^{-1/2}\|^2)^{1/2})$.

Proof. Let $x = (x_1, x_2) \in \mathcal{D}(\mathbf{a}) = \mathcal{G}_{1/2} \oplus \mathcal{G}$ and let $\mathbf{r}(x) = \dfrac{\mathbf{b}(x)}{\mathbf{a}(x)}$. Then

$$\mathbf{r}(x) = \frac{\|qx_1\|^2 + 2\,\mathrm{Re}(Vx_1|x_2)}{\|H^{1/2}x_1\|^2 + \|x_2\|^2}\ .$$

Set $y_1 = H^{1/2}x_1, x_2 = y_2$. Note that the mapping $x \mapsto y = (y_1, y_2)$ is a bijection of $\mathcal{D}(\mathbf{a})$ onto $\mathcal{G} \oplus \mathcal{G}$. Then

$$\mathbf{r}(y) = \frac{\|qH^{-1/2}y_1\|^2 + 2\,\mathrm{Re}(VH^{-1/2}y_1|y_2)}{\|y_1\|^2 + \|y_2\|^2}\ ,$$

hence for every $\gamma > 0$

$$\mathbf{r}(y) \le \frac{(\|qH^{-1/2}\|^2 + \gamma\|VH^{-1/2}\|^2)\|y_1\|^2 + \frac{1}{\gamma}\|y_2\|^2}{\|y_1\|^2 + \|y_2\|^2}\ ,$$

$$\mathbf{r}(y) \ge \frac{(\|qH^{-1/2}\|^2 - \gamma\|VH^{-1/2}\|^2)\|y_1\|^2 - \frac{1}{\gamma}\|y_2\|^2}{\|y_1\|^2 + \|y_2\|^2}\ .$$

Picking first

$$\gamma = \frac{-\|qH^{-1/2}\|^2 + (\|qH^{-1/2}\|^4 + 4\|VH^{-1/2}\|^2)^{1/2}}{2\|VH^{-1/2}\|^2}$$

and then

$$\gamma = \frac{\|qH^{-1/2}\|^2 + (\|qH^{-1/2}\|^4 + 4\|VH^{-1/2}\|^2)^{1/2}}{2\|VH^{-1/2}\|^2} ,$$

we find $\alpha \leq \mathbf{r}(y) \leq \beta$. □

Corollary 2.3. *There exist numbers $\kappa^- < 0 < \kappa^+$ such that for $\kappa \in (\kappa^-, \kappa^+)$ the form $\mathbf{a}_\kappa = \mathbf{a} + \kappa\mathbf{b}$ defined on $\mathcal{G}_{1/2} \oplus \mathcal{G}$ is closed, symmetric and bounded from below.*

Let $\mathbf{A}_\kappa$ be the associated operator in the Krein space $(\mathcal{H}, [\,\cdot\,|\,\cdot\,])$. From Lemmas 2.1, 2.2, and a result of P. Jonas (see [3, Proposition 6]) it follows that $\mathbf{A}_\kappa$ is a positive operator with nonempty resolvent set. From Theorem 1.1 (iii) and Lemma 2.1 we conclude (compare also to [2, Theorem 3.5])

Theorem 2.4. *Let q and V be symmetric $H^{1/2}-$ bounded operators in $\mathcal{G}$. Then for real κ with $|\kappa|$ sufficiently small, the operator $\mathbf{A}_\kappa$ is similar to a selfadjoint operator in the Hilbert space $(\mathcal{H}, (\,\cdot\,|\,\cdot\,))$.*

It follows from Theorem 2.4 that the operator $i\mathbf{A}_\kappa$ generates a uniformly bounded C_0 group of operators in $\mathcal{H}$. Since the Cauchy problem

$$\left(\frac{d}{dt} - i\kappa V\right)^2 u + (H + \kappa q^2)u = 0 \ , \ u(0) = u_0 \ , \ \frac{du}{dt}(0) = u_1 \ , \tag{2.3}$$

can be written as

$$\frac{dU}{dt} = i\mathbf{A}_\kappa U, \quad U(0) = U_0, \quad \text{where} \quad U = \begin{bmatrix} u \\ \left(-i\frac{d}{dt} - \kappa V\right) u \end{bmatrix} .$$

it follows that the Cauchy problem (2.3) is well-posed in $\mathcal{G}_{1/4} \oplus \mathcal{G}_{-1/4}$.

In particular, if $\mathcal{G} = L^2(\mathbb{R}^n)$ and H is the selfadjoint realization of the Laplace operator in $\mathcal{G}$, then we obtain a well-posedness result for the perturbed wave equation in $\mathbb{R}^n$. Note that the boundedness of $fH^{-1/2}$ in this case amounts to the inequality

$$\int_{\mathbb{R}^n} |fu|^2 \leq \int_{\mathbb{R}^n} |\nabla u|^2 \quad \text{for all} \quad u \in C_o^\infty(\mathbb{R}^n) \ .$$

We refer to the inequality (IV.4.6) in [7] for sufficient conditions to satisfy this inequality.

3. Elliptic operators with mildly varying coefficients on $\mathbb{R}^n$

For simplicity, we consider only the second order operators. Consider form

$$a_1(x,y) = \sum_{|\alpha|+|\beta|\le 2} \int a_{\alpha\beta} D^\alpha x D^\beta y$$

on $\mathcal{D}(a_1) = H^1(\mathbb{R}^n)$ where we assume

(i) for all α, β we have $a_{\alpha\beta} \in L^\infty(\mathbb{R}^n)$

(ii) For some weakly mixed elliptic polynomial (see [1])

$$p(x) = \sum_{|\alpha|+|\beta|\le 2} a^o_{\alpha\beta} x^{\alpha+\beta} \quad \text{we have} \quad \max_{\alpha,\beta}\{\|a_{\alpha\beta} - a^o_{\alpha\beta}\|_{L^\infty(\mathbb{R}^n)}\} < r \ ,$$

where r is a number to be specified later.

We are interested in the operator $A_1 = (\operatorname{sgn} x_n) \sum_{|\alpha|+|\beta|\le 2} D^\beta a_{\alpha\beta} D^\alpha$. Denote

$$a(x,y) = \sum_{|\alpha|+|\beta|\le 2} \int a^o_{\alpha\beta} D^\alpha x D^\beta y \ , b(x,y) = \sum_{|\alpha|+|\beta|\le 2} \int (a_{\alpha\beta} - a^o_{\alpha\beta}) D^\alpha x D^\beta y$$

for $x, y \in \mathcal{D}(a) = H^1(\mathbb{R}^n)$.

From the ellipticity of p we find the constant $C > 0$ such that $a(x) \ge C\|u\|^2_{H^1(\mathbb{R}^n)}$. From our assumption $|b(x)| \le Kra(x), x \in \mathcal{D}(a)$ with K dependent on a only. Hence for r sufficiently small, all our assumptions are satisfied. Since the operator JA, $J = \operatorname{sgn} x_n$, is similar to a selfadjoint operator in $L^2(\mathbb{R}^n)$ by [1], if r is sufficiently small the operator A_1 has the same property.

Similar results hold for the operator $A = \dfrac{1}{w} L$ where L is a second order positive elliptic operator in $L^2(\mathbb{R}^n)$ (or, more generally, in $L^2(\Omega)$ with appropriate boundary conditions) with constant coefficients and w is a function which vanishes on a set of measure zero and which attains positive and negative values on sets of positive measure. With its natural domain, the operator A is a selfadjoint operator in the Krein space L^2_w. We consider the perturbations of the form $B = \dfrac{1}{w} q$, where q is a function. Then the forms a and b are given by

$$a(x,y) = (Lx, y), \qquad b(x,y) = \Big((\operatorname{sgn} q)|q|^{1/2} x, |q|^{1/2} y \Big) \ .$$

Then $\dfrac{b}{a}$ is bounded if and only if $\dfrac{\||q|^{1/2}x\|^2_{L^2}}{a(x)}$ is bounded. This is equivalent to $|q|^{1/2} L^{-1/2}$ being a bounded operator in L^2. Sufficient conditions for that can be found in the literature; sufficient conditions for the boundedness of this operator in the case of finite domain Ω can be found in [4].

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