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ON SINGULAR CRITICAL POINTS OF POSITIVE OPERATORS IN KREIN SPACES

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ABSTRACT. We give an example of a positive operator B in a Krein space with the following properties: the nonzero spectrum of B consists of isolated simple eigenvalues, the norms of the orthogonal spectral projections in the Krein space onto the eigenspaces of B are uniformly bounded and the point ∞ is a singular critical point of B.

An operator A in the Krein space $(\mathcal{K}, [\,\cdot\,,\,\cdot\,])$ is said to be *positive* if $[Ax, x] > 0$ for all nonzero x in the domain of A. A bounded positive operator A in the Krein space $(\mathcal{K}, [\,\cdot\,,\,\cdot\,])$ has a projection valued spectral function E with 0 being its only possible critical point (see [1, Theorem IV.1.5] or [5, Section II.3.]). Recall that, by [5, Proposition 5.6], the condition

$$\|E((-\infty, \alpha])\| \le C_- < \infty \text{ for all } \alpha < 0 \tag{1}$$

is equivalent to the existence of the limit $\lim_{\alpha\uparrow 0} E((-\infty, \alpha])$ in the strong operator topology. Similarly,

$$\|E([\beta, \infty))\| \le C_+ < \infty \text{ for all } \beta > 0 \tag{2}$$

is equivalent to the existence of the limit $\lim_{\beta\downarrow 0} E([\beta, +\infty))$ in the strong operator topology. Since 0 is not an eigenvalue of a positive operator A, [5, Proposition 3.2] implies that (1) and (2) are equivalent. Also, if 0 is a critical point, it is said to be *regular* if one of the conditions (1) or (2) is fulfilled. If the critical point 0 is not regular, it is called *singular*.

In the sequel the operator A considered will have a discrete spectrum outside 0. Examples of bounded positive operators in $\mathcal{K}$ having 0 as a singular critical point can be constructed as follows (see also the examples in [2, Section 1], [3], [4]). Consider a sequence of two-dimensional Krein spaces $\mathcal{K}_n = \mathbb{C}^2$ with fundamental symmetry $J_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and positive operators A_n in $\mathcal{K}_n$; denote by λ_n^+ (λ_n^-, respectively) its positive (negative, respectively) eigenvalues and by P_n^+ (P_n^-, respectively) the orthogonal (in $\mathcal{K}_n$) projection onto the corresponding eigenspace.

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If A_n is chosen such that $\|A_n\| \le C$ for all n, $\lambda_n^+ \downarrow 0$, $\lambda_n^- \uparrow 0$, $\|P_n^\pm\| \to \infty$ if $n \to \infty$, then $A = \bigoplus_{n=1}^{\infty} A_n$ is a bounded positive operator in $\mathcal{K} = \bigoplus_{n=1}^{\infty} \mathcal{K}_n$ having 0 as a singular critical point. Evidently,

$$\sigma(A) = \{\lambda_n^+, \lambda_n^- | n \in \mathbb{N}\} \cup \{0\},$$

and $\|E(\{\lambda_n^\pm\})\| \to \infty$ if $n \to \infty$, that is, the eigenvectors f_n^+, f_n^- of A corresponding to λ_n^+ and λ_n^-, respectively, become arbitrarily close if n is large.

The question arises whether or not 0 can be a singular critical point of a positive operator A in $\mathcal{K}$ with discrete spectrum $\{\lambda_n^+, \lambda_n^- | n \in \mathbb{N}\}$ in $\mathbb{C} \setminus \{0\}$ if the projections $E(\{\lambda_n^\pm\})$ are uniformly bounded. It is the aim of this note to show that the answer is yes: We will construct a bounded positive operator A in a Krein space $\mathcal{K}$, such that the projections $E(\{\lambda_n^\pm\})$ corresponding to the single eigenvalues are uniformly bounded but, nevertheless,

$$\|E(\{\lambda_1^\pm, \dots, \lambda_n^\pm\})\| \longrightarrow \infty, \quad n \longrightarrow \infty \ .$$

Our construction is based on the following two lemmas.

Lemma 1. *Let $\mathcal{H}_n$ be an n-dimensional vector space with a positive definite scalar product $(\cdot\,,\cdot)$. Then there exist a basis $f_{n1}, \dots, f_{nn}$ of $\mathcal{H}_n$ and a positive contraction S_n in $\mathcal{H}_n$ such that*

$$0 < 1 \le \|f_{nk}\| \le 2, \quad \|S_n^{-1}\| = n, \quad (S_n f_{nj}, f_{nk}) = \delta_{jk}, \ j,k = 1, \dots, n \ .$$

Proof. Let $e_{n1}, \dots, e_{nn}$ be an orthonormal basis of $\mathcal{H}_n$, let T_n be the selfadjoint transformation in $\mathcal{H}_n$ given by $T_n e_{n1} = \sqrt{n} e_{n1}$, $T_n e_{nj} = e_{nj}$, $j = 2, \dots, n$, and put $S_n = T_n^{-2}$. Evidently, S_n is a positive selfadjoint contraction in $\mathcal{H}_n$, and $\min \sigma(S_n) = 1/n$. Therefore $\|S_n^{-1}\| = n$. Let $(u_{k1} \ \dots \ u_{kn})$, $k = 1, \dots, n$, be an orthonormal basis of the n-dimensional space of row vectors with components in $\mathbb{C}$, such that $u_{1j} = 1/\sqrt{n}$, $j = 1, \dots, n$. Then $U = (u_{kj})_{k,j=1}^n$ is a unitary matrix with $u_{1j} = 1/\sqrt{n}$, $j = 1, \dots, n$. Put

$$\phi_{nj} = \sum_{k=1}^{n} u_{kj} e_{nk}, \quad j = 1, \dots, n \ .$$

Then ϕ_{nj}, $j = 1, \dots, n$, is an orthonormal basis of $\mathcal{H}_n$ and

$$\|T_n \phi_{nj}\|^2 = n\frac{1}{n} + \sum_{k=2}^{n} |u_{kj}|^2 = 1 + 1 - \frac{1}{n}, \ j = 1, \dots, n \ .$$

Hence $1 \le \|T_n \phi_{nj}\| \le 2$. Let $f_{nj} = T_n \phi_{nj}$, $j = 1, \dots, n$. Then $1 \le \|f_{nj}\| \le 2$ and $(S_n f_{nj}, f_{nk}) = (\phi_{nj}, \phi_{nk}) = \delta_{jk}$, $j,k = 1, \dots n$. The lemma is proved. □

Lemma 2. *Let $(\mathcal{H}, (\cdot\,,\cdot))$ be a separable Hilbert space and let P be a positive, bounded and boundedly invertible operator in $\mathcal{H}$. Let ϕ_j, $j \in \mathbb{N}$, be a Riesz basis of $\mathcal{H}$ such that $(P\phi_j, \phi_k) = \delta_{jk}$, $j,k \in \mathbb{N}$, and let $\lambda_j \in \mathbb{C}$, $j \in \mathbb{N}$, be a bounded sequence. Define the operator A in $\mathcal{H}$ by $A\phi_j = \lambda_j \phi_j$, $j \in \mathbb{N}$. Then, A can be extended by continuity to a bounded linear operator in $\mathcal{H}$ such that $\|A\| \le \sqrt{\|P\|\|P^{-1}\|} \sup\{|\lambda_j|, \ j \in \mathbb{N}\}$.*

Proof. For a bounded and boundedly invertible positive operator P we have

$$(3) \qquad \|P^{-1}\|^{-1}(x,x) \le (Px,x) \le \|P\|(x,x), \quad x \in \mathcal{H} \ .$$

Since the vectors $\phi_j, j \in \mathbb{N}$, are orthonormal with respect to the inner product $(P\,\cdot, \cdot)$, it follows that

$$(PAx, Ax) \le (\sup\{|\lambda_j|, j \in \mathbb{N}\})^2 (Px, x), \quad x \in \mathcal{H}\ . \tag{4}$$

Combining (3) and (4) we get

$$\begin{aligned}\|Ax\|^2 = (Ax, Ax) &\le \|P^{-1}\|(PAx, Ax) \le \|P^{-1}\|(\sup\{|\lambda_j|, j \in \mathbb{N}\})^2(Px, x)\\ &\le \|P^{-1}\|\|P\|(\sup\{|\lambda_j|,\ j \in \mathbb{N}\})^2\|x\|^2\end{aligned}$$

and the lemma follows. □

Theorem. *There exist a Krein space* $(\mathcal{K}, [\,\cdot\,,\,\cdot\,])$ *and a bounded positive operator* A *in* $\mathcal{K}$ *with the following properties:*

(a) *The nonzero spectrum of* A *consists of isolated simple eigenvalues.*
(b) *The point* 0 *is a singular critical point of* A.
(c) *The norms of the orthogonal projections in the Krein space* $(\mathcal{K}, [\,\cdot\,,\,\cdot\,])$ *onto the eigenspaces of* A *are uniformly bounded.*

Proof. With the notation as in Lemma 1, choose $\mathcal{H}_n^+ = \mathcal{H}_n^- = \mathcal{H}_n$. Let $\mathcal{K}_n = \mathcal{H}_n^+ \oplus \mathcal{H}_n^-$ be the direct sum of the Hilbert spaces $(\mathcal{H}_n^\pm, (\,\cdot\,,\,\cdot\,))$. The positive definite inner product on $\mathcal{K}_n$ is also denoted by $(\,\cdot\,,\,\cdot\,)$. All norms in $\mathcal{K}_n$ correspond to this inner product. Endow $\mathcal{K}_n = \mathcal{H}_n^+ \oplus \mathcal{H}_n^-$ with the indefinite inner product $[\,\cdot\,,\,\cdot\,]$ given by the fundamental symmetry $J_n = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$. Consider the operator $K_n^+ = (I_n - S_n)^{1/2}$ acting from $\mathcal{H}_n^+$ into $\mathcal{H}_n^-$ as an angular operator in $\mathcal{K}_n$. Here S_n is the operator constructed in Lemma 1. Let $\mathcal{L}_n^+$ be the graph of K_n^+ in $\mathcal{K}_n = \mathcal{H}_n^+ \oplus \mathcal{H}_n^-$. Then $\mathcal{L}_n^+$ is an n-dimensional maximal positive subspace in $\mathcal{K}_n$. It is spanned by the vectors $\boldsymbol{f}_{nk}^+ = \begin{pmatrix} f_{nk} \\ K_n^+ f_{nk} \end{pmatrix}$, $k = 1, \ldots, n$, and

$$[\boldsymbol{f}_{nk}^+, \boldsymbol{f}_{nj}^+] = (f_{nk}, f_{nj}) - (K_n^+ f_{nk}, K_n^+ f_{nj}) = (S_n f_{nk}, f_{nj}) = \delta_{kj}\ , \tag{5}$$

$$\|\boldsymbol{f}_{nk}^+\|^2 = \|f_{nk}\|^2 + \|K_n^+ f_{nk}\|^2 \le 2\|f_{nk}\|^2 \le 8\ . \tag{6}$$

Denote by $\mathcal{L}_n^-$ the orthogonal complement of $\mathcal{L}_n^+$ in the Krein space $\mathcal{K}_n$. Then $\mathcal{L}_n^-$ is a maximal negative subspace of $\mathcal{K}_n$. The operator $K_n^- = (I_n - S_n)^{1/2}$, acting from $\mathcal{H}_n^-$ into $\mathcal{H}_n^+$, is the angular operator of $\mathcal{L}_n^-$. The subspace $\mathcal{L}_n^-$ is spanned by the vectors $\boldsymbol{f}_{nk}^- = \begin{pmatrix} K_n^- f_{nk} \\ f_{nk} \end{pmatrix}$, $k = 1, \ldots, n$. This follows from the linear independence of $f_{n1}, \ldots, f_{nn}$ and the relation

$$\begin{aligned}[\boldsymbol{f}_{nj}^+, \boldsymbol{f}_{nk}^-] &= (f_{nj}, K_n^- f_{nk}) - (K_n^+ f_{nj}, f_{nk})\\ &= (f_{nj}, (I - S_n)^{1/2} f_{nk}) - ((I - S_n)^{1/2} f_{nj}, f_{nk}) = 0\ .\end{aligned} \tag{7}$$

The decomposition $\mathcal{K}_n = \mathcal{L}_n^+ [\dot{+}] \mathcal{L}_n^-$ is a fundamental decomposition of $(\mathcal{K}_n, [\,\cdot\,,\,\cdot\,])$. Solving a corresponding system of vector equations we find that the orthogonal (fundamental) projections $Q_n^\pm$ of the Krein space $\mathcal{K}_n$ onto $\mathcal{L}_n^\pm$ are given by

$$Q_n^+ = \begin{pmatrix} I_n \\ K_n^+ \end{pmatrix} S_n^{-1} \begin{pmatrix} I_n & -K_n^- \end{pmatrix}\ , \quad Q_n^- = \begin{pmatrix} K_n^- \\ I_n \end{pmatrix} S_n^{-1} \begin{pmatrix} -K_n^+ & I_n \end{pmatrix}.$$

From Lemma 1 it follows that $\|S_n^{-1}\| = n$. This and the above matrix representations of $Q_n^\pm$ imply that

$$n \le \|Q_n^\pm\| \le 2n\ . \tag{8}$$

Consequently, for any $\boldsymbol{f} \in \mathcal{K}_n$ we have

$$\|Q_n^{\pm}\boldsymbol{f}\| \le 2n\|\boldsymbol{f}\| \ .$$

It follows from (5) that the vectors $\boldsymbol{f}_{n1}^+, \dots, \boldsymbol{f}_{nn}^+$ form an orthonormal basis in the Hilbert space $(\mathcal{L}_n^+, [\,\cdot\,,\,\cdot\,])$. Denote by

$$P_{nk}^+ = \frac{[\,\cdot\,, \boldsymbol{f}_{nk}^+]}{[\boldsymbol{f}_{nk}^+, \boldsymbol{f}_{nk}^+]}\boldsymbol{f}_{nk}^+, \quad k = 1, \dots, n \ ,$$

the orthogonal projection in the Krein space $\mathcal{K}_n$ onto the subspace spanned by the vector $\boldsymbol{f}_{nk}^+, k = 1, \dots, n$. Then, by (5) and (6),

$$1 \le \|P_{nk}^+\| = \frac{\|\boldsymbol{f}_{nk}^+\|^2}{[\boldsymbol{f}_{nk}^+, \boldsymbol{f}_{nk}^+]} \le 8, \quad k = 1, \dots, n \ . \tag{9}$$

Further, the operator

$$J_{n1} := Q_n^+ - Q_n^-$$

is a fundamental symmetry in $(\mathcal{K}_n, [\,\cdot\,,\,\cdot\,])$. In particular, the inner product

$$(\boldsymbol{x}, \boldsymbol{y})_1 := [J_{n1}\boldsymbol{x}, \boldsymbol{y}], \ \boldsymbol{x}, \boldsymbol{y} \in \mathcal{K}_n,$$

is positive definite. Therefore, the operator $J_n J_{n1}$ is positive and invertible in the Hilbert space $(\mathcal{K}_n, (\,\cdot\,,\,\cdot\,))$. Note also that $J_{n1} = J_{n1}^{-1}$. It follows from (8) that $\|J_{n1}\| = \|J_{n1}^{-1}\| \le \|Q_n^+\| + \|Q_n^-\| \le 4n$. Consequently,

$$\|J_n J_{n1}\| = \|(J_n J_{n1})^{-1}\| \le 4n \ . \tag{10}$$

The vectors $\boldsymbol{f}_{nj}^+, \boldsymbol{f}_{nk}^-, \ j,k = 1, \dots, n$, are orthonormal in $(\mathcal{K}_n, (\,\cdot\,,\,\cdot\,)_1)$. This follows from (5), (7) and the relation

$$(\boldsymbol{f}_{nj}^+, \boldsymbol{f}_{nk}^-)_1 = [(Q_n^+ - Q_n^-)\boldsymbol{f}_{nj}^+, \boldsymbol{f}_{nk}^-] = [Q_n^+\boldsymbol{f}_{nj}^+, \boldsymbol{f}_{nk}^-] = [\boldsymbol{f}_{nj}^+, \boldsymbol{f}_{nk}^-] = 0 \ .$$

Now we can apply Lemma 2 to the vectors $\boldsymbol{f}_{nj}^+, \ \boldsymbol{f}_{nk}^-, \ j,k = 1, \dots, n$, and the positive operator $J_n J_{n1}$: For given $\lambda_1^{\pm}, \dots, \lambda_n^{\pm} \in \mathbb{C}$ define an operator A_n by

$$A_n \boldsymbol{f}_{nj}^{\pm} = \lambda_{nj}^{\pm}\boldsymbol{f}_{nj}^{\pm}, \ j = 1, \dots, n,$$

and then extend it by linearity to $\mathcal{K}_n$. It follows from Lemma 2 and (10) that

$$\|A_n\| \le 4n \max\{|\lambda_j^{\pm}|, \ j = 1, \dots, n\} \le 4C. \tag{11}$$

Let $\mathcal{K}$ be the Krein space which is the direct orthogonal sum of the Krein spaces $\mathcal{K}_n, \ n \in \mathbb{N}$,

$$\mathcal{K} := \bigoplus_{n=1}^{\infty} \mathcal{K}_n \ .$$

The vectors $\boldsymbol{f}_{nj}^{\pm}, \ j = 1, \dots, n, \ n \in \mathbb{N}$, constructed above are considered as elements of $\mathcal{K}$ and the Krein spaces $\mathcal{K}_n, \ n \in \mathbb{N}$, are considered as mutually orthogonal subspaces of $\mathcal{K}$. The vectors $\boldsymbol{f}_{nj}^{\pm}, \ j = 1, \dots, n$, form a basis for $\mathcal{K}_n$. Let $\lambda_{nj}^{\pm}, \ j = 1, \dots, n$, be distinct real numbers such that $\pm\lambda_{nj}^{\pm} > 0, \ j = 1, \dots, n$, and such that there exists a constant C with

$$n \max\{|\lambda_{nj}^{\pm}|, \ j = 1, \dots, n\} \le C \tag{12}$$

for all $n \in \mathbb{N}$.

Put

$$A := \bigoplus_{n=1}^{\infty} A_n \ .$$

Then A is a positive operator in the Krein space $(\mathcal{K}, [\,\cdot\,,\,\cdot\,])$, and from (11) and (12) we get $\|A\| \leq 4C$. Since the linear span of the vectors $\boldsymbol{f}_{nj}^{\pm},\ j = 1, \ldots, n,\ n \in \mathbb{N}$, is dense in $\mathcal{K}$, it follows from the spectral theorem (see [1, Theorem IV.1.5] or [5, Theorem 3.1]) that the nonzero spectrum of A consists of the simple eigenvalues $\lambda_{nj}^{\pm},\ j = 1, \ldots, n,\ n \in \mathbb{N}$. Consequently, the left-hand side of the inequality (8) implies that 0 is a singular critical point of A and the right-hand side of the inequality (9) implies that the norms of the orthogonal projections in $(\mathcal{K}, [\,\cdot\,,\,\cdot\,])$ onto the eigenspaces of A are uniformly bounded by 8. The theorem is proved. □

Remark. We can arrange the numbers $\lambda_{nj}^{+},\ j = 1, \ldots, n,\ n \in \mathbb{N}$, in a lower triangular table. Also, we can put the sequence $\{\frac{1}{m},\ m \in \mathbb{N}\}$ in a lower triangular table by ending each row with a triangular number $\frac{n(n+1)}{2}$ in the denominator. A comparison of these two tables leads to

$$\lambda_{nj}^{\pm} := \pm \left(\frac{n(n-1)}{2} + j \right)^{-1}, \quad j = 1, \ldots, n,\ n \in \mathbb{N} \ . \tag{13}$$

In this way we get

$$\left\{\lambda_{nj}^{\pm},\ j = 1, \ldots, n,\ n \in \mathbb{N}\right\} = \left\{ \pm \frac{1}{m},\ m \in \mathbb{N} \right\} \ .$$

The numbers $\lambda_{nj}^{\pm}$ in (13) satisfy (12) with $C = 2$. The proof of the Theorem implies that the nonzero spectrum of the operator A, which was constructed by means of the numbers $\lambda_{nj}^{\pm}$ from (13), consists of the simple eigenvalues $\pm\frac{1}{m},\ m \in \mathbb{N}$.

If we consider the inverse $B = A^{-1}$ of the operator A from the previous theorem and with the specific choice of numbers $\lambda_{nj}^{\pm}$ as in the Remark, we get:

Corollary. *There exist a Krein space $(\mathcal{K}, [\,\cdot\,,\,\cdot\,])$ and an unbounded positive operator B in $\mathcal{K}$ with the following properties:*

(a) *The nonzero spectrum of B consists of isolated simple eigenvalues.*
(b) *The point ∞ is a singular critical point of B.*
(c) *For each positive number μ we have*

$$\|E([a,b))\| \leq 8\lfloor \mu \rfloor \quad \textit{whenever} \quad b - a < \mu \ ,$$

where E is the spectral function of B and $\lfloor \mu \rfloor$ denotes the largest integer smaller than μ.

Proof. Let A be the operator defined in the proof of the Theorem with the specific choice of the numbers $\pm\lambda_{nj}$ as in the Remark. Then $B = A^{-1}$ is a positive operator with a nonempty resolvent set (see e.g. [5, Proposition 3.1]), and $\sigma(B) = \mathbb{Z} \setminus \{0\}$. Let $\mu > 0$ be arbitrary and let $0 < b - a < \mu$. Then the interval $[a,b)$ contains at most $\lfloor \mu \rfloor$ eigenvalues of B. Therefore, $\|E([a,b))\| \leq 8\lfloor \mu \rfloor$. □

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