# INDEFINITE STURM-LIOUVILLE OPERATORS IN POLAR FORM

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ABSTRACT. We consider the indefinite Sturm-Liouville differential expression

$$\mathfrak{a}(f) := -\frac{1}{w}\left(\frac{1}{r}f'\right)',$$

where $\mathfrak{a}$ is defined on a finite or infinite open interval $I$ with $0 \in I$ and the coefficients $r$ and $w$ which are locally summable and such that $r(x)$ and $(\operatorname{sgn} x)w(x)$ are positive a.e. on $I$. With the differential expression $\mathfrak{a}$ we associate a non-negative self-adjoint operator $A$ in the Krein space $L_w^2(I)$ which is viewed as a coupling of symmetric operators in Hilbert spaces related to the intersections of $I$ with the positive and the negative semi-axis. For the operator $A$ we derive conditions in terms of the coefficients $w$ and $r$ for the existence of a Riesz basis consisting of generalized eigenfunctions of $A$ and for the similarity of $A$ to a self-adjoint operator in a Hilbert space $L_{|w|}^2(I)$. These results are obtained as consequences of abstract results about the regularity of critical points of nonnegative self-adjoint operators in Krein spaces which are couplings of two symmetric operators acting in Hilbert spaces.

## CONTENTS

## 1. Introduction

Let $I = (b_-, b_+)$ be a finite or infinite interval such that $-\infty \le b_- < 0 < b_+ \le +\infty$. We consider the indefinite Sturm-Liouville differential expression $\mathfrak{a}$ on $I$ that is given in polar form

$$\big(\mathfrak{a}f\big)(x) := -\frac{1}{w(x)}\frac{d}{dx}\left(\frac{1}{r(x)}\frac{d}{dx}f(x)\right), \tag{1.1}$$

where the coefficients $r$ and $w$ are real functions on $I$ satisfying the conditions

$$r, w \in L^1_{\mathrm{loc}}(I) \quad \text{and} \quad r(x), \ (\mathrm{sgn}\,x)w(x) > 0 \quad \text{for a.a.} \quad x \in I. \tag{1.2}$$

With the differential expression $\mathfrak{a}$ we associate a closed linear operator $A$ in the weighted Hilbert space space $L^2_{|w|}(I)$. The operator $A$ is not self-adjoint in $L^2_{|w|}(I)$ but it is self-adjoint and nonnegative in the Krein space $L^2_w(I)$ which coincides with $L^2_{|w|}(I)$ as a normed vector space and has indefinite inner product

$$[f, g]_w := \int_I f(x)\overline{g(x)}w(x)dx,$$

see [22] for a similar setting.

We are interested in the following two properties of the differential operator $A$:

- (Ri) *Riesz basis property*, that is, the existence of a Riesz basis of the Hilbert space $L^2_{|w|}(I)$ which consists of eigenfunctions and generalized eigenfunctions of $A$;
- (Si) *Similarity* of $A$ to a self-adjoint operator in the Hilbert space $L^2_{|w|}(I)$, that is, the existence of a bounded and boundedly invertible operator $T$ such that $T^{-1}AT$ is self-adjoint in the Hilbert space $L^2_{|w|}(I)$.

Our results will be formulated in terms of the functions

$$W_\pm(x) := \int_0^x w_\pm(\xi)d\xi, \quad R_\pm(x) := \int_0^x r_\pm(\xi)d\xi, \quad x \in I_\pm, \tag{1.3}$$

where $I_- = (b_-, 0)$, $I_+ = (0, b_+)$, $w_-$ is the restriction of $-w$ onto $I_-$ and $w_+$ is the restriction of $w$ onto $I_+$.

The first result of this kind was given by Beals in [5]. The first proof of the existence of a weight $w$, with $r = 1$, for which $A$ does not have the Riesz basis property was given by Volkmer in [76]. The first characterization of the Riesz basis property for the operator $A$ was given by Parfenov [61] in the case when $w$ is odd and $r = 1$. Using Pyatkov's approach via interpolation spaces [64, 65], Parfenov proves that the Riesz basis property for the operator $A$ holds if and only if the

function $W_+$ is positively increasing at $0_+$. Recall that a nondecreasing positive function $\varphi$ is called *positively increasing at* $0_+$ if there exists $\lambda \in (0,1)$ such that $\limsup_{x\downarrow 0}\big(\varphi(\lambda x)/\varphi(x)\big) < 1$; $\psi$ is *positively increasing at* $0_-$ if $x \mapsto -\psi(-x)$ is positively increasing at $0_+$.

In [58] Kostenko used a different method to characterize the properties (Si) and (Ri) for differential operator $A$ with odd $w$ and even $r$. In particular, it was shown in [58] that the Riesz basis property for the operator $A$ holds if and only if the function $W_+ \circ R_+^{-1}$ is positively increasing at $0_+$.

One of the main results of this paper is the following theorem in which we give a sufficient condition for the Riesz basis property, in the spirit of Parfenov's and Kostenko's results, but without the assumptions that $w$ is odd and $r$ is even. We also give a new kind of a characterization of the Riesz basis property when $W_\pm \circ R_\pm^{-1}$ are slowly varying functions. Recall that a measurable positive function $\varphi$ is said to be *slowly varying at* $0_+$ if for all $\lambda > 0$ we have $\lim_{x\downarrow 0}\big(\varphi(\lambda x)/\varphi(x)\big) = 1$; $\psi$ is *slowly varying at* $0_-$ if $x \mapsto -\psi(-x)$ is slowly varying at $0_+$, for more about slowly varying functions see Appendix A.

**Theorem A.** *Let the differential expression* $\mathfrak{a}$ *satisfy* (1.2) *and let* $W_\pm$ *and* $R_\pm$ *be the functions defined in* (1.3). *Assume that the spectrum of the operator* $A$ *associated with the differential expression* $\mathfrak{a}$ *in the Hilbert space* $L^2_{|w|}(I)$ *is discrete. Then the eigenvalues of* $A$ *accumulate on both sides of* $\infty$ *and the following two statements hold.*

(a) *If either* $W_+ \circ R_+^{-1}$ *is positively increasing at* $0_+$ *or* $W_- \circ R_-^{-1}$ *is positively increasing at* $0_-$, *then the operator* $A$ *has the Riesz basis property* (Ri).

(b) *If* $W_+ \circ R_+^{-1}$ *is slowly varying at* $0_+$ *and* $W_- \circ R_-^{-1}$ *is slowly varying at* $0_-$, *then the Riesz basis property* (Ri) *is equivalent to the condition*

$$\left(1 + \frac{W_-\big(R_-^{-1}(-x)\big)}{W_+\big(R_+^{-1}(x)\big)}\right)^{-1} = O(1) \quad as \quad x \downarrow 0. \tag{1.4}$$

The main tool that we use in this paper is Langer's spectral theory of definitizable operators in Krein spaces, see [59]. Our differential operator $A$ is a nonnegative self-adjoint operator with a nonempty resolvent set in the Krein space $L^2_w(I)$. This is a special kind of a definitizable operator that admits a spectral function $E$ defined on $\mathbb{R} \setminus \{0\}$ which behaves similarly to the spectral function of a self-adjoint operator in a Hilbert space with a possible exception of two critical points 0 and $\infty$. A critical point is called regular if $E_t$ is bounded in a neighbourhood of that point. Otherwise, a critical point is called singular. By $c_s(A) \subseteq \{0,\infty\}$ we denote the set of singular critical points of $A$ and by $c_r(A) \subseteq \{0,\infty\}$ the set of regular critical points of $A$.

In the case of discrete spectrum of the differential operator $A$, the Riesz basis property of $A$ is equivalent to the regularity of the critical point $\infty$, see [16, Proposition 4.1]. This fact and the paper of Beals [5] were motivation for [16, 18, 33, 76, 8, 9, 34, 57, 15] to study definitizability and the regularity of the critical point infinity for differential operators; see also a detailed survey by Fleige [35].

The regularity of both critical points of $A$ is equivalent to $A$ being similar to self-adjoint operator in a Hilbert space. This fact was used by Ćurgus and Najman in [17] to prove that the operator associated with (1.1) where $w(x) = \mathrm{sgn}(x)$, $r = 1$ and $I = \mathbb{R}$ is similar to a self-adjoint operator in the Hilbert space $L^2(\mathbb{R})$. This

result was reproved and generalized by Krein space and other methods by several authors, see [19, 20, 36, 48, 32, 49, 50, 52, 58].

The questions of similarity of a differential operator to self-adjoint operator and the existence of a Riesz basis consisting of its eigenfunctions arise in problems of numerical computation of eigenvalues. For example, in [39, Subsection 4.1.2] the authors study the differential expression (1.1) with $w(x) = x^3$, $r = 1$ and $I = [-1, 1]$. To construct an efficient and accurate eigensolver for the associated differential operator it was important that the operator is similar to a self-adjoint operator and that its eigenfunctions form a Riesz basis of the Hilbert space $L^2_{|w|}[-1, 1]$. This offers a potential application of our results in numerical computation of eigenvalues for differential operators with more general coefficients than considered in [39].

This paper is organized as follows. In Sections 2 and 3 we establish conditions for the regularity of the critical points 0 and $\infty$ for a nonnegative self-adjoint operator $A$ with a nonempty resolvent set in an abstract Krein space $\mathcal{K}$. We use a boundary triple approach to extension theory developed in [55, 37, 26] and we construct $A$ as a coupling of two abstract symmetric operators $A_+ = B_+$ and $A_- = -B_-$, where $B_+$ and $B_-$ are nonnegative symmetric operators with defect numbers $(1, 1)$ acting in Hilbert spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ which form a fundamental decomposition for $\mathcal{K}$. The coupling method was developed for operators in Hilbert spaces in [25] and it was used in [48], and also in [51, 49, 50, 58], to study the problem of similarity of differential operators with indefinite weights to self-adjoint operators in Hilbert spaces. When boundary triples $\left(\mathbb{C}, \Gamma_0^+, \Gamma_1^+\right)$ and $\left(\mathbb{C}, \Gamma_0^-, \Gamma_1^-\right)$ for the operators $B_+$ and $B_-$ are fixed the coupling $A$ of the operators $A_+$ and $A_-$ relative to these boundary triples is uniquely defined as a self-adjoint operator acting in the Krein space $\mathcal{K}$ with the fundamental decomposition $\mathcal{K} = \mathcal{H}_+[+]\mathcal{H}_-$, see Theorem 3.1.

We use the resolvent criterion of K. Veselić [75] to study regularity of critical points of the coupling $A$ in terms of the Weyl functions $m_+$ and $m_-$ of the operators $B_+$ and $B_-$. It was shown in [51, 49] that the so-called $D_\infty$-property

$$\frac{\max\left\{\operatorname{Im} m_+(iy), \operatorname{Im} m_-(iy)\right\}}{\left|m_+(iy) + m_-(-iy)\right|} = O(1) \quad \text{as} \quad y \to +\infty$$

is necessary for $\infty \notin c_s(A)$. In Theorem 3.10 we prove that $D_\infty$-property is also sufficient for $\infty \notin c_s(A)$ provided that the Weyl functions $m_+$ and $m_-$ satisfy the condition:

$$\text{For some } y_0 > 0 \quad \operatorname{Re} m_+(iy) \operatorname{Re} m_-(iy) > 0 \quad \text{for all} \quad y > y_0. \tag{1.5}$$

Moreover, in Theorem 3.13 we prove that under the assumption (1.5) the one-sided condition

$$\operatorname{Im} m_+(iy) = O(\operatorname{Re} m_+(iy)) \quad \text{as} \quad y \to +\infty$$

is sufficient for $\infty \notin c_s(A)$. In Theorem 3.11 and Theorem 3.14 we prove analogous results for $0 \notin c_s(A)$.

In Section 4, the abstract results from Section 3 are adapted to indefinite Sturm-Liouville operators. Let $\mathcal{H}_\pm$ be the weighted spaces $\mathcal{H}_\pm := L^2_{w_\pm}(I_\pm)$ and let $B_\pm$ be nonnegative symmetric operators generated in $\mathcal{H}_\pm$ by the differential expressions

$$\mathfrak{b}_\pm(f) := -\frac{1}{w_\pm}\left(\frac{1}{r_\pm}f'\right)' \quad \text{on} \quad I_\pm. \tag{1.6}$$

Using the above scheme we represent the operator $A$ as a coupling of two symmetric operators $A_+ := B_+$ and $A_- := -B_-$. Conditions for regularity of critical points

0 and $\infty$ of the differential operator are formulated in terms of the functions (1.3). We use the results of Bennewitz [7] and Kostenko [58] to reformulate one-sided sufficient conditions for regularity of critical points $\infty$ or 0 from Theorem 3.13 in terms of the functions $W_{\pm}$ and $R_{\pm}$. Specifically, in Theorem 4.13, we show that if either $W_+ \circ R_+^{-1}$ is positively increasing at $0_+$ or $W_- \circ R_-^{-1}$ is positively increasing at $0_-$, then $\infty$ is a regular critical point for the operator $A$ associated with indefinite differential expression (1.1). In Theorem 4.17 we prove that in the case of slowly varying functions $W_{\pm} \circ R_{\pm}^{-1}$ the condition $\infty \in c_r(A)$ is equivalent to the condition (1.4) in Theorem A. This result is obtained by combining the characterization of the regularity of the critical point $\infty$ for the operator $A$ from Theorem 3.10 with the Atkinson asymptotic formula for the Weyl functions $m_+(iy)$ and $m_-(iy)$ proved in [3] and [7]. Theorem A is a special case of Theorems 4.13 and 4.17.

To show the strength of our results, in Example 4.21 we present an indefinite Sturm-Liouville operator $A$ with a nonodd weight for which Theorem 4.17 guarantees that $\infty$ is a regular critical point, but other known criteria for regularity such as Volkmer's condition from [76], Fleige's condition from [15], Parfenov's condition [62] cannot be applied.

In Theorem 4.28 we give a list of sufficient conditions under which we have $0 \notin c_s(A)$ for the differential operator $A$. In particular, it is shown that in the case when $w_+ \in L_1(I_+)$ and $w_- \in L_1(I_-)$ the following equivalence holds

$$0 \notin c_s(A) \quad \text{and} \quad \ker A = \ker A^2 \quad \Leftrightarrow \quad W_+(b_+) + W_-(b_-) \neq 0. \qquad (1.7)$$

The proof of this theorem is based on abstract results from Theorems 3.11 and 3.14 and asymptotic formulas for the Weyl functions of the operators $B_+$ and $B_-$ from Lemmas 4.10 and 4.11.

In Theorem 4.34 we combine the regularity results for the points 0 and $\infty$ to obtain new results about similarity of the operator $A$ to a self-adjoint operator in a Hilbert space. In the particular case when $w_+ \in L_1(I_+)$ and $w_- \in L_1(I_-)$ these results take the folowing form

**Theorem B.** *Let the differential expression $\mathfrak{a}$ satisfy (1.2) and let $W_{\pm}$, $R_{\pm}$ be the functions defined in (1.3). Assume that $w_+ \in L_1(I_+)$, $w_- \in L_1(I_-)$ and one of the equivalent conditions in (1.7) is satisfied. Then the following statements hold.*

- (i) *If either $W_+ \circ R_+^{-1}$ is positively increasing at $0_+$ or $W_- \circ R_-^{-1}$ is positively increasing at $0_-$, then (Si) holds for $A$.*
- (ii) *If $W_+ \circ R_+^{-1}$ is slowly varying at $0_+$ and $W_- \circ R_-^{-1}$ is slowly varying at $0_-$, then similarity property (Si) for $A$ is equivalent to condition (1.4).*

In Section 4 we systematically use results of Karamata theory, see for example [56], about positively increasing and slowly varying functions which are presented and developed for our purposes in Appendix A. In particular, it is shown that the condition for the function $W_{\pm} \circ R_{\pm}^{-1}$ to be slowly varying is equivalent to Atkinson condition (4.19), see Corollary A.8.

1.1. **Notation.** By $\mathbb{C}$ we denote the set of complex numbers and by $\mathbb{R}$ the set of real numbers. By $\mathbb{C}_+$ (resp. $\mathbb{C}_-$) we denote the set of all $z \in \mathbb{C}$ with positive (resp. negative) imaginary part. Similarly, $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) stands for the set of all positive (resp. negative) reals. For $z \in \mathbb{C}$, $\overline{z}$, $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the complex conjugate, real and imaginary part of $z$.

All operators in this paper are closed densely defined linear operators. For such an operator $T$, we use the common notation $\rho(T)$, $\mathrm{dom}(T)$, $\mathrm{ran}(T)$ and $\ker(T)$ for the resolvent set, the domain, the range and the null-space, respectively, of $T$.

We use the asymptotic notation little-$o$, big-$O$ and $\sim$ defined at $+\infty$ as follows: $f(x) = o\big(g(x)\big)$ as $x \to +\infty$ if and only if $\lim_{x \to +\infty} f(x)/g(x) = 0$; $f(x) = O\big(g(x)\big)$ as $x \to +\infty$ if and only if there exist $M, a \in \mathbb{R}_+$ such that $|f(x)| \leq M|g(x)|$ for all $x \geq a$; $f(x) \sim g(x)$ as $x \to +\infty$ if and only if $\lim_{x \to +\infty} f(x)/g(x) = 1$. Similar notation is used in the right and left neighborhood of $0$ with analogous definitions.

## 2. Preliminaries

2.1. **Definitizable operators in Krein spaces.** A Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ is a complex vector space $\mathcal{K}$ with a sesquilinear form $[\,\cdot\,,\cdot\,]_\mathcal{K}$ such that there exist subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ of $\mathcal{K}$ with $\big(\mathcal{H}_+, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ and $\big(\mathcal{H}_-, -[\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ being Hilbert spaces and $\mathcal{K} = \mathcal{H}_+[\dot{+}]\mathcal{H}_-$ is a direct and orthogonal sum; this direct orthogonal sum is called a *fundamental decomposition* of a Krein space $\mathcal{K}$. Let $P_+$ and $P_-$ be projections associated with the direct sum $\mathcal{K} = \mathcal{H}_+\dot{+}\mathcal{H}_-$. The operator $J := P_+ - P_-$ is called a *fundamental symmetry* of a Krein space. The space $\mathcal{K}$ with the inner product $\langle x, y\rangle_\mathcal{K} = [Jx, y]_\mathcal{K}$, $x, y \in \mathcal{K}$, is a Hilbert space. All topological notions in a Krein space refer to the topology of the Hilbert space $\big(\mathcal{K}, \langle\,\cdot\,,\cdot\,\rangle_\mathcal{K}\big)$. For the general theory of Krein spaces and operators acting in them we refer to the monographs [4, 11]. For a subspace $\mathcal{L} \subset \mathcal{K}$ denote by $\kappa_+(\mathcal{L})$ (resp. $\kappa_-(\mathcal{L})$) the least upper bound of the dimensions of positive (resp. negative) subspaces of $\mathcal{L}$.

Let $A$ be a linear operator in a Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ with a dense domain $\mathrm{dom}\, A$. The *adjoint* of $A$ with respect to the inner product $[\,\cdot\,,\cdot\,]_\mathcal{K}$ is denoted by $A^{[*]}$. The operator $A$ is called *symmetric* in $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ if $A^{[*]}$ is an extension of $A$ and $A$ is called *self-adjoint* in $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ if $A = A^{[*]}$. The operator $A$ is called *nonnegative* in $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ if $[Af, f]_\mathcal{K} \geq 0$ for all $f \in \mathrm{dom}\, A$.

According to [59] a self-adjoint operator $A$ is called *definitizable*, if its resolvent set $\rho(A)$ is nonempty and there exists a real polynomial $p$ such that $p(A)$ is nonnegative. Such polynomial $p$ is called definitizing polynomial of $A$. The non-real spectrum of a definitizable operator consists of a finite set of points symmetric with respect to $\mathbb{R}$. A real number $\lambda$ is said to be a *critical point* of $A$ if $p(\lambda) = 0$ for every definitizing polynomial $p$ of $A$. Similarly, $\infty$ is a *critical point* of $A$, if at least one of its definitizing polynomials $p$ is of odd degree and the real spectrum of $A$ is neither bounded from below, nor bounded from above. The set of critical points of $A$ is denoted by $c(A)$.

In particular, a self-adjoint nonnegative operator $A$ with nonempty resolvent set $\rho(A)$ is definitizable with definitizing polynomial $p(\lambda) = \lambda$ and the only possible critical points of $A$ are $0$ and $\infty$.

A definitizable operator $A$ admits a spectral function $E$, see [59, Theorem II.3.1], defined on the semiring $\mathcal{R}$ generated by all intervals whose endpoints are not critical points of $A$ with $E(\Delta)$ being self-adjoint projection in $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ for every $\Delta \in \mathcal{R}$. Moreover,

$$\big(E(\Delta)\mathcal{K}, [\,\cdot\,,\cdot\,]_\mathcal{K}\big) \text{ is a Hilbert space whenever } \Delta \subset \{t \in \mathbb{R} : p(t) > 0\}. \qquad (2.1)$$

It follows from the properties of the spectral function $E$, see [59], that the restriction of $A$ to its spectral subspace $E(\Delta)\mathcal{K}$ in (2.1) is a self-adjoint operator in the Hilbert space $\big(E(\Delta)\mathcal{K}, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$. A similar statement holds for intervals in $\{t \in \mathbb{R} : p(t) < 0\}$.

However, if one of the endpoints of the interval approaches a critical point, it may happen that the norms of the corresponding spectral projections are unbounded. More precisely, a point $\alpha \in c(A)$ is called a *regular critical point* of $A$, if there exists a neighbourhood $G$ of $\alpha$ such that

the set of projections $\quad \{E(\Delta) : \Delta \in \mathcal{R}, \ \overline{\Delta} \subset G \setminus \{\alpha\}\} \quad$ is bounded.

The set of all regular critical points of $A$ is denoted by $c_r(A)$. A critical point of $A$ which is not regular is called *singular critical point* of $A$. The set of all singular critical points of $A$ is denoted by $c_s(A)$. It is often difficult to decide whether a critical point is singular or regular. A widely used characterization for $\infty \notin c_s(A)$ is from K. Veselić [75], see also [41, Corollaries 1.5 and 1.6], [2]. Due to the Uniform Boundedness Principle it can be reformulated as follows.

**Theorem 2.1.** *Let $A$ be a definitizable operator in a Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$ and $\alpha \in \mathbb{R}$. Then:*

(a) $\infty \notin c_s(A)$ *if and only if there exists $\eta_0 > 0$ such that for every $f \in \mathcal{K}$*

$$\int_{\eta_0}^{\eta} \mathrm{Re}\big[(A - iy)^{-1}f, f\big]_{\mathcal{K}} dy = O(1) \quad as \quad \eta \to +\infty. \tag{2.2}$$

(b) $\alpha \notin c_s(A)$ *and $\ker(A - \alpha) = \ker\big((A - \alpha)^2\big)$ if and only if there exists $\eta_0 > 0$ such that for every $f \in \mathcal{K}$*

$$\int_{\eta}^{\eta_0} \mathrm{Re}\big[(A - \alpha - iy)^{-1}f, f\big]_{\mathcal{K}} dy = O(1) \quad as \quad \eta \to 0. \tag{2.3}$$

Let us consider a nonnegative operator $A$ in a Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$. Then, as mentioned above, the only possible critical points are 0 and $\infty$. Assume that (2.2) holds. Then, by the proof of [40, Lemma 1] this implies that the set of projections $E((1,n))$ and $E((-n,-1))$, $n \in \mathbb{N}$, is bounded, which in turn implies $\infty \notin c_s(A)$. Moreover, it is easy to see that the new inner product defined for $f, g \in (I - E([-1,1]))\mathcal{K}$ via

$$\langle f, g \rangle_{\mathrm{new}} := \lim_{n \to \infty} [(E(1,n)) - E((-n,-1)f, g]$$

is a Hilbert space inner product and that the restriction of $A$ to $(I - E([-1,1]))\mathcal{K}$ is self-adjoint in the corresponding Hilbert space. A similar reasoning using (2.3) holds for the point zero. This implies the following well-known statement (see, e.g. [59]).

**Theorem 2.2.** *A nonnegative operator $A$ in a Krein space has the similarity property (Si) if and only if $\rho(A) \neq \emptyset$, $\ker(A) = \ker(A^2)$ and $0, \infty \notin c_s(A)$.*

2.2. **Boundary triples and Weyl functions of symmetric operators.** In this subsection we will use $S$ for a closed densely defined symmetric operator in the Krein space $(\mathcal{K}, [\,\cdot\,,\cdot\,]_{\mathcal{K}})$. Let $\widehat{\rho}(S)$ denote the set of points of regular type of $S$, see [1], and let $\mathfrak{N}_z$ denote the defect subspace of the operator $S$

$$\mathfrak{N}_z := \mathrm{ran}(S - \overline{z})^{[\perp]}, \quad z \in \widehat{\rho}(S).$$

The numbers $n_{\pm}(S) := \dim(\mathfrak{N}_z)$ are constant for all $z \in \widehat{\rho}(S) \cap \mathbb{C}_{\pm}$ and are called defect numbers of $S$.

In what follows we assume that the operator $S$ admits a self-adjoint extension $\widetilde{S}$ in $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ with a nonempty resolvent set $\rho(\widetilde{S})$. Then for all $z \in \rho(\widetilde{S})$ we have

$$\mathrm{dom}(S^{[*]}) = \mathrm{dom}(\widetilde{S}) \dotplus \mathfrak{N}_z \quad \text{direct sum in} \quad \mathcal{H}. \tag{2.4}$$

This implies, in particular, that the dimension $\dim(\mathfrak{N}_z)$ is constant for all $z \in \rho(\widetilde{S})$ and hence $n_+(S) = n_-(S)$. Moreover, we assume everywhere in this paper that $n_\pm(S) = 1$. Notice that the coincidence of defect numbers of a symmetric operator $S$ in a Krein space does not imply the existence of self-adjoint extension $\widetilde{S}$ of $S$, see [70].

**Definition 2.3.** Let $\Gamma_0$ and $\Gamma_1$ be linear mappings from $\mathrm{dom}(S^{[*]})$ to $\mathbb{C}$ such that

(i) the mapping $\Gamma : f \to \begin{pmatrix} \Gamma_0 f \\ \Gamma_1 f \end{pmatrix}$ from $\mathrm{dom}(S^{[*]})$ to $\mathbb{C}^2$ is surjective;

(ii) the abstract Green's identity

$$\big[S^{[*]}f, g\big]_\mathcal{K} - \big[f, S^{[*]}g\big]_\mathcal{K} = (\Gamma_1 f)\overline{(\Gamma_0 g)} - (\Gamma_0 f)\overline{(\Gamma_1 g)} \tag{2.5}$$

holds for all $f, g \in \mathrm{dom}(S^{[*]})$.

Then the triple $\big(\mathbb{C}, \Gamma_0, \Gamma_1\big)$ is called a *boundary triple* for $S^+$, see [37, 26, 23] for much more general setting.

It follows from (2.5) that the extensions $S_0$, $S_1$ of $S$ defined as restrictions of $S^+$ to the domains

$$\mathrm{dom}(S_0) := \ker(\Gamma_0), \quad \text{and} \quad \mathrm{dom}(S_1) := \ker(\Gamma_1)$$

are self-adjoint extensions of $S$.

Given a self-adjoint extension $\widetilde{S}$ of $S$ with nonempty $\rho(\widetilde{S})$ one can always choose a boundary triple $\big(\mathbb{C}, \Gamma_0, \Gamma_1\big)$ for $S$ such that $S_0 = \widetilde{S}$, see [24, Proposition 2.2]. In this case for every $z \in \rho(S_0)$ the decomposition (2.4) holds with $\widetilde{S} = S_0$ and the mapping $\Gamma_0|_{\mathfrak{N}_z} : \mathfrak{N}_z \to \mathbb{C}$ is invertible for every $z \in \rho(S_0)$. A vector-valued function $\gamma(z)$ defined on $\rho(S_0)$ with values in $\mathfrak{N}_z$ is called the $\gamma$-field of $S$, associated with the boundary triple $\big(\mathbb{C}, \Gamma_0, \Gamma_1\big)$ if

$$\Gamma_0 \gamma(z) = 1 \quad \text{for all} \quad z \in \rho(S_0).$$

Notice, that $\gamma(z)$ satisfies the equality, see [24, Proposition 2.2],

$$\gamma(z) = (S_0 - z_0)(S_0 - z)^{-1}\gamma(z_0), \quad z, z_0 \in \rho(S_0) \tag{2.6}$$

and hence the vector-valued function $\gamma(z)$ is holomorphic on $\rho(S_0)$.

**Definition 2.4.** The function $M(z)$ defined by the equality

$$M(z)\Gamma_0 f_z = \Gamma_1 f_z, \quad f_z \in \mathfrak{N}_z, z \in \rho(S_0),$$

is called the *abstract Weyl function* of $S$, corresponding to the boundary triple $\big(\mathbb{C}, \Gamma_0, \Gamma_1\big)$.

The notion of the abstract Weyl function was introduced in [26] for a Hilbert space symmetric operator and in [23] for a Krein space symmetric operator.

Clearly, $M(z) = \Gamma_1 \gamma(z)$ for $z \in \rho(S_0)$, and hence $M(z)$ is well defined. It follows from (2.5) and (2.6) that the Weyl function $M(z)$ satisfies the identity

$$M(z) - \overline{M(w)} = (z - \overline{w})\,[\gamma(z), \gamma(w)]_\mathcal{K}, \quad z, w \in \rho(S_0). \tag{2.7}$$

With $w = \overline{z}$ the identity (2.7) yields that the Weyl function $M$ satisfies the symmetry condition

$$M(\overline{z}) = \overline{M(z)}, \quad z \in \rho(S_0). \tag{2.8}$$

In the case when $\big(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}}\big)$ is a Hilbert space we will use the notation $B$ for a closed densely defined symmetric operator in the Hilbert space $\mathcal{H}$ with defect numbers $(1, 1)$. Let $\big(\mathbb{C}, \Gamma_0, \Gamma_1\big)$ be a boundary triple for $B^{\langle * \rangle}$. We will use the notations $m(z)$ and $\gamma_B(z)$ for the abstract Weyl function and for the $\gamma$-field of $B$ corresponding to the boundary triple $\big(\mathbb{C}, \Gamma_0, \Gamma_1\big)$. It follows from (2.7) and (2.8) that $m$ is a Nevanlinna function, see [45], i.e. $m$ is holomorphic at least on $\mathbb{C} \setminus \mathbb{R}$ and satisfies the following two conditions

$$m(\overline{z}) = \overline{m(z)} \quad \text{and} \quad \operatorname{Im} m(z) \geq 0, \qquad z \in \mathbb{C}_+.$$

Since the operator $B$ is densely defined the following two conditions hold (see [27, Theorem 7.36])

$$\lim_{y \uparrow +\infty} y^{-1} m(iy) = 0, \quad \lim_{y \uparrow +\infty} y \operatorname{Im} m(iy) = +\infty. \tag{2.9}$$

Assume that the operators $B$ and its self-adjoint extension $B_0$ with the domain $\operatorname{dom} B_0 = \ker \Gamma_0$ are nonnegative. Then the Weyl function $m$ is holomorphic on $\mathbb{R}_-$. A Nevanlinna function $m$ with the above property which, in addition, takes nonnegative values for all $z \in \mathbb{R}_-$ is called a Stieltjes function. The class of all Stieltjes functions is denoted by $\mathcal{S}$.

A Stieltjes function $m$ admits the integral representation, [45],

$$m(z) = a + \int_0^{+\infty} \frac{d\sigma(t)}{t - z} \tag{2.10}$$

with $a \geq 0$ and with a non-decreasing function $\sigma(t)$, such that $\int_0^{+\infty} \frac{d\sigma(t)}{1+t}$ converges Notice, that for the Stieltjes function $m$ with the integral representation (2.10) the latter condition in (2.9) is equivalent to

$$\int_0^{+\infty} d\sigma(t) = +\infty. \tag{2.11}$$

The following statement is immediate from (2.10) and (2.11).

**Proposition 2.5.** *Let $m \in \mathcal{S}$ and (2.9) hold. Then $\operatorname{Re} m(iy) > 0$ for all $y \in \mathbb{R}_+$.*

2.3. **Real operators.** Recall the notions of real operator and real vector valued function with respect to some conjugation, see [30, Section III.5] and [26, 55].

**Definition 2.6.** An involution $j_{\mathcal{K}}$ on a Krein space $\big(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}}\big)$ is called a *conjugation* on $\mathcal{K}$ if

$$[j_{\mathcal{K}} f, j_{\mathcal{K}} g]_{\mathcal{K}} = [g, f]_{\mathcal{K}} \quad \text{for all} \quad f, g \in \mathcal{K}. \tag{2.12}$$

A closed operator $T$ in a Krein space $\mathcal{K}$ is called *real*, if

$$j_{\mathcal{K}} \operatorname{dom}(T) = \operatorname{dom}(T) \quad \text{and} \quad j_{\mathcal{K}} T = T j_{\mathcal{K}}.$$

Every conjugation is an anti-linear operator, see [73, Section IX.2], i.e.

$$j_{\mathcal{K}}(\lambda f + \mu g) = \overline{\lambda} j_{\mathcal{K}} f + \overline{\mu} j_{\mathcal{K}} g \quad \text{for all} \quad f, g \in \mathcal{K}, \ \lambda, \mu \in \mathbb{C}.$$

If $T$ is real and densely defined then its adjoint $T^{[*]}$ is also a real operator in $\mathcal{K}$.

A vector $f$ in $\mathcal{K}$ is called *real* with respect to the conjugation $j_\mathcal{K}$, if $j_\mathcal{K} f = f$. An arbitrary vector $f \in \mathcal{K}$ can be decomposed into the sum

$$f = f^R + i f^I, \quad \text{where} \quad f^R = \frac{1}{2}(f + j_\mathcal{K} f) \quad \text{and} \quad f^I = \frac{1}{2i}(f - j_\mathcal{K} f) \quad \text{are real.}$$
$$(2.13)$$

Let $j$ be the standard conjugation in $\mathbb{C}$, $jz = \overline{z}$ for all $z \in \mathbb{C}$. A scalar function $m(z)$ is called *real*, if its domain of holomorphy $\mathfrak{h}_m$ is symmetric with respect to $\mathbb{R}$ and $m(\overline{z}) = \overline{m(z)}$ for all $z \in \mathfrak{h}_m$. Similarly, a vector valued function $\gamma(z)$ with domain of holomorphy $\mathfrak{h}_\gamma = \overline{\mathfrak{h}_\gamma}$ and values in $\mathcal{K}$, $\gamma : \mathfrak{h}_\gamma \to \mathcal{K}$, is called *real*, if

$$\gamma(\overline{z}) = j_\mathcal{K} \gamma(z) \quad \text{for all} \quad z \in \mathfrak{h}_\gamma. \tag{2.14}$$

Let a symmetric operator $S$ be real in $\mathcal{K}$ with the conjugation $j_\mathcal{K}$. A boundary triple $(\mathbb{C}, \Gamma_0, \Gamma_1)$ for $S^{[*]}$ is called *real*, if

$$j\Gamma_0 = \Gamma_0 j_\mathcal{K} \quad \text{and} \quad j\Gamma_1 = \Gamma_1 j_\mathcal{K}.$$

Every real symmetric operator $S$ admits a real boundary triple $(\mathbb{C}, \Gamma_0, \Gamma_1)$ and the corresponding Weyl function $M$ and the $\gamma$-field $\gamma$ are real, see [55] for the case of a Hilbert space $\mathcal{K}$.

## 3. REGULARITY OF CRITICAL POINTS OF COUPLINGS IN KREIN SPACES

3.1. **Couplings of symmetric operators in Krein spaces.** In this section we consider two Krein spaces $(\mathcal{K}_+, [\cdot, \cdot]_{\mathcal{K}_+})$ and $(\mathcal{K}_-, [\cdot, \cdot]_{\mathcal{K}_-})$. Let their direct sum

$$\mathcal{K} = \mathcal{K}_+ [\dot{+}] \mathcal{K}_-$$

be endowed with the natural inner product

$$[f_+ + f_-, g_+ + g_-]_\mathcal{K} := [f_+, g_+]_{\mathcal{K}_+} + [f_-, g_-]_{\mathcal{K}_-}, \quad f_\pm, g_\pm \in \mathcal{K}_\pm. \tag{3.1}$$

Consider two closed symmetric densely defined operators $A_+$ and $A_-$ with defect numbers $(1,1)$ acting in the Krein spaces $(\mathcal{K}_+, [\cdot, \cdot]_{\mathcal{K}_+})$ and $(\mathcal{K}_-, [\cdot, \cdot]_{\mathcal{K}_-})$. Let $(\mathbb{C}, \Gamma_0^\pm, \Gamma_1^\pm)$ be a boundary triple for $A_\pm^{[*]}$. Let $M_\pm$ and $\gamma_{A_\pm}$ be the corresponding Weyl function and the $\gamma$-field. By $A_{\pm,0}$ we denote the self-adjoint extension of $A_\pm$ which is defined on

$$\mathrm{dom}(A_{\pm,0}) = \ker(\Gamma_0^\pm) \quad \text{by} \quad A_{\pm,0} = A_\pm^{[*]}\big|_{\ker(\Gamma_0^\pm)}.$$

Then the functions $M_\pm$ are defined and holomorphic on $\rho(A_{\pm,0})$. Assume that

$$\rho(A_{+,0}) \cap \rho(A_{-,0}) \neq \emptyset. \tag{3.2}$$

The following theorem is an indefinite version of a result from [25] which is, in this form, presented in [14, 29].

**Theorem 3.1.** *Let $A_\pm$ be closed symmetric densely defined operators with defect numbers $(1,1)$ in the Krein spaces $\mathcal{K}_\pm$. Let $(\mathbb{C}, \Gamma_0^\pm, \Gamma_1^\pm)$ be boundary triples for $A_\pm^{[*]}$ which satisfy (3.2). Let $M_\pm$ and $\gamma_{A_\pm}$ be the Weyl functions and the $\gamma$-fields of $A_\pm$ corresponding to the boundary triples $(\mathbb{C}, \Gamma_0^\pm, \Gamma_1^\pm)$, and let $S$ and $A$ be the restrictions of $A_+^{[*]}[+]A_-^{[*]}$ to the domains*

$$\mathrm{dom}(S) = \left\{ \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : \begin{array}{l} \Gamma_0^+(f_+) = \Gamma_0^-(f_-) = 0, \\ \Gamma_1^+(f_+) + \Gamma_1^-(f_-) = 0, \end{array} \quad f_\pm \in \mathrm{dom}(A_\pm^{[*]}) \right\}, \tag{3.3}$$

$$\mathrm{dom}(A) = \left\{ \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : \begin{array}{l} \Gamma_0^+(f_+) = \Gamma_0^-(f_-), \\ \Gamma_1^+(f_+) + \Gamma_1^-(f_-) = 0, \end{array} \quad f_\pm \in \mathrm{dom}\big(A_\pm^{[*]}\big) \right\}. \qquad (3.4)$$

*Then the following statements hold:*

(a) *The operator $S$ is symmetric with defect numbers $(1,1)$ and $A$ is a self-adjoint extension of $S$.*

(b) *The adjoint $S^+$ of $S$ is the restriction of $A_+^+[+]A_-^+$ to the domain*

$$\mathrm{dom}(S^+) = \left\{ \begin{pmatrix} f_+ \\ f_- \end{pmatrix} : \Gamma_0^+(f_+) = \Gamma_0^-(f_-), \ f_\pm \in \mathrm{dom}\big(A_\pm^{[*]}\big) \right\}.$$

*and a boundary triple $\big(\mathbb{C}, \Gamma_0, \Gamma_1\big)$ for $S^{[*]}$ is given by*

$$\Gamma_0 f = \Gamma_0^+ f_+, \quad \Gamma_1 f = \Gamma_1^+ f_+ + \Gamma_1^- f_-, \quad f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in \mathrm{dom}\big(S^{[*]}\big). \qquad (3.5)$$

(c) *The Weyl function and the $\gamma$-field of $S$ relative to the boundary triple $\big(\mathbb{C}, \Gamma_0, \Gamma_1\big)$ are*

$$M(z) = M_+(z) + M_-(z), \quad \gamma(z) = \begin{pmatrix} \gamma_{A_+}(z) \\ \gamma_{A_-}(z) \end{pmatrix}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \qquad (3.6)$$

(d) *If $z \in \rho(A_{+,0}) \cap \rho(A_{-,0})$ then $z \in \rho(A)$ if and only if $M_+(z) + M_-(z) \neq 0$.*

(e) *The resolvent of the operator $A$ is given by*

$$\big(A - z\big)^{-1} f = \big(A_0 - z\big)^{-1} f - \frac{[f, \gamma(\overline{z})]_{\mathcal{K}}}{M_+(z) + M_-(z)} \gamma(z), \quad z \in \rho(A) \cap \rho(A_0), \qquad (3.7)$$

*where $A_0 = A_{+,0}[+]A_{-,0}$ and $f \in \mathcal{K}$.*

**Definition 3.2.** The operator $A$ defined in Theorem 3.1(a) is called the *coupling of the operators $A_+$ and $A_-$ in the Krein space $\big(\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}}\big)$ relative to the triples* $\big(\mathbb{C}, \Gamma_0^+, \Gamma_1^+\big)$ *and* $\big(\mathbb{C}, \Gamma_0^-, \Gamma_1^-\big)$ *and $A_0 = A_{+,0}[+]A_{-,0}$ is called the* *decoupled* *operator.*

The next statement was proved in [58, Lemma 5.4]. For the convenience of the reader we present here a proof based on Theorem 3.1.

**Lemma 3.3.** *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ be a Hilbert space with a conjugation $j_{\mathcal{H}}$, let $B$ be a closed densely defined real symmetric operator in $\mathcal{H}$ with defect numbers $(1,1)$, let $\big(\mathbb{C}, \Gamma_0, \Gamma_1\big)$ be a real boundary triple for $B^{\langle * \rangle}$, let $m$ and $\gamma_B$ be the corresponding Weyl function and the $\gamma$-field for $B$ and define*

$$\widehat{h}(z) = \big\langle h, \gamma_B(\overline{z}) \big\rangle_{\mathcal{H}}, \quad h \in \mathcal{H}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \qquad (3.8)$$

*Then the following inequality holds for all real $h \in \mathcal{H}$:*

$$\int_0^\infty \frac{\big|\mathrm{Im}\,\widehat{h}(iy)^2\big|}{\mathrm{Im}\,m(iy)} dy \leq 2\pi \|h\|_{\mathcal{H}}^2, \qquad (3.9)$$

*Proof.* Let $\mathcal{K}_+$ and $\mathcal{K}_-$ be two copies of the Hilbert space $\mathcal{H}$ and let us set $A_+ := B$ and $A_- := -B$. Notice that $\big(\mathbb{C}, \Gamma_0, \Gamma_1\big)$ is a boundary triple for $A_+^{[*]}$, $\big(\mathbb{C}, \Gamma_0, -\Gamma_1\big)$ is a boundary triple for $A_-^+$ and the corresponding Weyl functions $M_+(z)$, $M_-(z)$ and the $\gamma$-fields $\gamma_{A_+}(z)$ and $\gamma_{A_-}(z)$ take the form

$$M_+(z) = m(z), \quad M_-(z) = -m(-z), \quad \gamma_{A_+}(z) = \gamma_B(z), \quad \gamma_{A_-}(z) = \gamma_B(-z).$$

Let $A$ be the coupling of $A_+$, $A_-$ acting in the Hilbert space $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_- = \mathcal{H} \oplus \mathcal{H}$, let $A_0 = B_0 \oplus (-B_0)$ be the decoupled operator as defined in Definition 3.2, $B_0$

being the restriction of $B^{\langle * \rangle}$ to $\ker \Gamma_0$ and let us denote by $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ the scalar product in $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_- = \mathcal{H} \oplus \mathcal{H}$. Applying Theorem 3.1 to the operators $A_+$ and $A_-$ one obtains from (3.7) for vector $f = h \oplus 0$, $h \in \mathcal{H}$ the equality

$$\langle (A - iy)^{-1} f, f \rangle_{\mathcal{K}} = \langle (A_0 - iy)^{-1} f, f \rangle_{\mathcal{K}} - \frac{\widehat{h}(iy)\overline{\widehat{h}(-iy)}}{m(iy) - m(-iy)} \tag{3.10}$$

Since $A$ and $A_0$ are self-adjoint operators in the Hilbert space $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$, an application of the functional calculus yields

$$\int_0^\infty \left| \mathrm{Re} \langle (A - iy)^{-1} f, f \rangle_{\mathcal{K}} \right| dy \leq \frac{\pi}{2} \|f\|_{\mathcal{K}}^2, \tag{3.11}$$

$$\int_0^\infty \left| \mathrm{Re} \langle (A_0 - iy)^{-1} f, f \rangle_{\mathcal{K}} \right| dy \leq \frac{\pi}{2} \|f\|_{\mathcal{K}}^2. \tag{3.12}$$

for all $f \in \mathcal{K}$. Since the boundary triple $(\mathbb{C}, \Gamma_0, \Gamma_1)$ is real, $\gamma_B$ is real as well. If, in addition, $h$ is real, then

$$j_{\mathcal{H}} h = h, \quad j_{\mathcal{H}} \gamma_B(iy) = \gamma_B(-iy) \quad \text{for all} \quad y \in \mathbb{R}_+$$

and by (3.8) and Definition 2.6

$$\overline{\widehat{h}(-iy)} = \langle \gamma_B(iy), h \rangle_{\mathcal{H}} = \langle j_{\mathcal{H}} h, j_{\mathcal{H}} \gamma_B(iy) \rangle_{\mathcal{H}} = \langle h, \gamma_B(-iy) \rangle_{\mathcal{H}} = \widehat{h}(iy). \tag{3.13}$$

By (3.10), (3.11), (3.12) and (3.13)

$$\int_0^\infty \left| \mathrm{Re} \frac{\widehat{h}(iy)^2}{m(iy) - m(-iy)} \right| dy \leq \pi \|f\|_{\mathcal{H}}^2 = \pi \|h\|_{\mathcal{H}}^2.$$

Using the equality $m(iy) - m(-iy) = 2i \, \mathrm{Im} \, m(iy)$ one obtains for all real $h \in \mathcal{H}$:

$$\int_0^\infty \frac{\left| \mathrm{Im} \, \widehat{h}(iy)^2 \right|}{2 \, \mathrm{Im} \, m(iy)} dy = \int_0^\infty \left| \mathrm{Re} \frac{\widehat{h}(iy)^2}{m(iy) - m(-iy)} \right| dy \leq \pi \|h\|_{\mathcal{H}}^2.$$

This proves (3.9). $\qquad \square$

In the following lemma we apply Theorem 3.1 to two real symmetric operators $B_+$ and $B_-$ acting in Hilbert spaces $\mathcal{H}_+$ and $\mathcal{H}_-$ and obtain estimates for a family of weighted $L^2$-norms of "generalized Fourier transforms"

$$\widehat{f}_\pm(z) = \langle f_\pm, \gamma_{B_\pm}(\overline{z}) \rangle_{\mathcal{H}_\pm}, \quad f_\pm \in \mathcal{H}_\pm, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{3.14}$$

**Lemma 3.4.** *Let $\mathcal{H}_\pm$ be Hilbert spaces with conjugations $j_{\mathcal{H}_\pm}$, let $B_\pm$ be closed densely defined real symmetric operators in $\mathcal{H}_\pm$ with defect numbers $(1, 1)$, let $(\mathbb{C}, \Gamma_0^\pm, \Gamma_1^\pm)$ be real boundary triples for $B_\pm^{\langle * \rangle}$, and let $m_\pm$ and $\gamma_{B_\pm}$ be the corresponding Weyl functions and the $\gamma$-fields for $B_\pm$. Then the following inequalities hold for all real $f_\pm \in \mathcal{H}_\pm$:*

$$\int_0^{+\infty} \left| \widehat{f}_\pm(iy) \right|^2 \frac{\left| \mathrm{Re}(m_+(iy) + m_-(iy)) \right|}{\left| m_+(iy) + m_-(iy) \right|^2} dy \leq 5\pi \|f_\pm\|_{\mathcal{H}_\pm}^2, \tag{3.15}$$

*Proof.* **1.** In this step we prove that for all real $f_\pm \in \mathcal{H}_\pm$ we have

$$\int_0^{+\infty} \left| \mathrm{Re} \, \widehat{f}_\pm(iy)^2 \right| \frac{\left| \mathrm{Re}(m_+(iy) + m_-(iy)) \right|}{\left| m_+(iy) + m_-(iy) \right|^2} dy \leq 3\pi \|f_\pm\|_{\mathcal{H}_\pm}^2. \tag{3.16}$$

Applying Theorem 3.1 to the operators $A_+ := B_+$ and $A_- := B_-$ in Hilbert spaces $\mathcal{K}_+ = \mathcal{H}_+$, $\mathcal{K}_- := \mathcal{H}_-$ and taking $f = f_+ \oplus f_-$, $f_\pm \in \mathcal{H}_\pm$, one obtains the equality

$$\big\langle (A - iy)^{-1} f, f \big\rangle = \big\langle (A_0 - iy)^{-1} f, f \big\rangle - \frac{(\widehat{f}_+(iy) + \widehat{f}_-(iy))\overline{(\widehat{f}_+(-iy) + \widehat{f}_-(-iy))}}{m_+(iy) + m_-(iy)}$$

where $A$ is the coupling of $A_+$ and $A_-$ defined by (3.4), $A_0$ is the decoupled operator, as defined in Definition 3.2. Since $A$ and $A_0$ are self-adjoint operators in the Hilbert space $\mathcal{H} := \mathcal{H}_+ \oplus \mathcal{H}_-$ one obtains from (3.11), (3.12) and (3.13) for all real $f_\pm \in \mathcal{H}_\pm$

$$\int_0^{+\infty} \left| \mathrm{Re} \, \frac{\big(\widehat{f}_+(iy) + \widehat{f}_-(iy)\big)^2}{m_+(iy) + m_-(iy)} \right| dy \le \pi \|f\|_{\mathcal{H}}^2. \tag{3.17}$$

Set

$$u_\pm(iy) := \mathrm{Re}\, m_\pm(iy), \quad v_\pm(iy) := \mathrm{Im}\, m_\pm(iy), \tag{3.18}$$

$$U(iy) := \mathrm{Re}\big((\widehat{f}_+(iy) + \widehat{f}_-(iy))^2\big), \quad V(iy) := \mathrm{Im}\big((\widehat{f}_+(iy) + \widehat{f}_-(iy))^2\big).$$

Then inequality (3.17) can be rewritten as

$$\int_0^{+\infty} \frac{\big|U(iy)\big(u_+(iy) + u_-(iy)\big) + V(iy)(v_+(iy) + v_-(iy))\big|}{\big|m_+(iy) + m_-(iy)\big|^2} dy \le \pi \|f\|_{\mathcal{H}}^2.$$

In particular, setting subsequently $f_- = 0$ or $f_+ = 0$, one obtains

$$\int_0^{+\infty} \frac{\left| \mathrm{Re}\big(\widehat{f}_\pm(iy)^2\big)\big(u_+(iy) + u_-(iy)\big) + \mathrm{Im}\big(\widehat{f}_\pm(iy)^2\big)\big(v_+(iy) + v_-(iy)\big) \right|}{\big|m_+(iy) + m_-(iy)\big|^2} dy$$

$$\le \pi \|f_\pm\|_{\mathcal{H}_\pm}^2. \tag{3.19}$$

By (3.9) for every real $f_\pm \in \mathcal{H}_\pm$

$$\int_0^{+\infty} \frac{\left| \mathrm{Im}\big(\widehat{f}_\pm(iy)^2\big)\big(v_+(iy) + v_-(iy)\big) \right|}{\big|m_+(iy) + m_-(iy)\big|^2} dy \le \int_0^{+\infty} \frac{\left| \mathrm{Im}\big(\widehat{f}_\pm(iy)^2\big) \right|}{\mathrm{Im}\, m_\pm(iy)} dy \le 2\pi \|f_\pm\|_{\mathcal{H}_\pm}^2 \tag{3.20}$$

and thus (3.19) and (3.20) imply for every real $f_\pm \in \mathcal{H}_\pm$

$$\int_0^{+\infty} \left| \mathrm{Re}\big(\widehat{f}_\pm(iy)^2\big) \right| \frac{|u_+(iy) + u_-(iy)|}{\big|m_+(iy) + m_-(iy)\big|^2} dy \le 3\pi \|f_\pm\|_{\mathcal{H}_\pm}^2,$$

which proves (3.16).

**2.** To prove (3.15) we notice that from

$$\frac{|u_+(iy) + u_-(iy)|}{|m_+(iy) + m_-(iy)|} \le 1$$

and (3.9) we obtain

$$\int_0^{+\infty} \left| \mathrm{Im}\big(\widehat{f}_\pm(iy)^2\big) \right| \frac{|u_+(iy) + u_-(iy)|}{\big|m_+(iy) + m_-(iy)\big|^2} dy \le \int_0^{+\infty} \frac{\left| \mathrm{Im}\big(\widehat{f}_\pm(iy)^2\big) \right|}{|m_+(iy) + m_-(iy)|} dy$$

$$\le \int_0^{+\infty} \frac{\left| \mathrm{Im}\big(\widehat{f}_\pm(iy)^2\big) \right|}{\mathrm{Im}\, m_\pm(iy)} dy \tag{3.21}$$

$$\le 2\pi \|f_\pm\|_{\mathcal{H}_\pm}^2,$$

for all real $f_\pm \in \mathcal{H}_\pm$. Now (3.15) follows from (3.16) and (3.21). $\qquad \square$

**Remark 3.5.** It follows from (3.13) that the inequalities (3.9) in Lemma 3.3 and (3.15) in Lemma 3.4 remain in force when we substitute $\widehat{f}_\pm(iy)$ by $\widehat{f}_\pm(-iy)$ for all real $f_\pm \in \mathcal{H}_\pm$:

$$\int_0^{+\infty} \frac{\left|\mathrm{Im}\big(\widehat{f}_\pm(-iy)^2\big)\right|}{\mathrm{Im}\, m_\pm(iy)} dy \;\le\; 2\pi \|f_\pm\|_{\mathcal{H}_\pm}^2,$$

$$\int_0^{+\infty} \left|\widehat{f}_\pm(-iy)\right|^2 \frac{|\mathrm{Re}(m_+(iy) + m_-(iy))|}{|m_+(iy) + m_-(iy)|^2} dy \;\le 5\pi \|f_\pm\|_{\mathcal{H}_\pm}^2.$$

3.2. **Veselić condition and coupling.** In the rest of this section we make the following general assumptions:

(A1) $\big(\mathcal{H}_\pm, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}_\pm}\big)$ are Hilbert spaces with conjugations $j_{\mathcal{H}_\pm}$ and $\big(\mathcal{K}_\pm, [\,\cdot\,,\cdot\,]_{\mathcal{K}_\pm}\big)$ are Krein spaces defined by

$$\mathcal{K}_+ = \mathcal{H}_+, \quad [\,\cdot\,,\cdot\,]_{\mathcal{K}_+} = \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}_+}, \quad \mathcal{K}_- = \mathcal{H}_-, \quad [\,\cdot\,,\cdot\,]_{\mathcal{K}_-} = -\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}_-}.$$

(A2) $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ is a Krein space with the fundamental decomposition $\mathcal{K} := \mathcal{K}_+[+]\mathcal{K}_-$ and the inner product (3.1).

(A3) $B_\pm$ are real closed nonnegative symmetric densely defined operators with defect indices $(1,1)$ in the Hilbert spaces $\big(\mathcal{H}_\pm, \langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}_\pm}\big)$ and let $A_+$ and $A_-$ be symmetric operators in the Krein spaces $\mathcal{K}_+$ and $\mathcal{K}_-$, respectively:

$$A_+ := B_+, \quad A_- := -B_-.$$

(A4) $\big(\mathbb{C}, \Gamma_0^\pm, \Gamma_1^\pm\big)$ are real boundary triples for $B_\pm^{\langle *\rangle}$ and $m_\pm(z)$ and $\gamma_{B_\pm}(z)$ are the corresponding Weyl functions and the $\gamma$−fields.

(A5) $A$ is the coupling of the operators $A_+$ and $A_-$ in the Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$ relative to the triples $\big(\mathbb{C}, \Gamma_0^+, \Gamma_1^+\big)$ and $\big(\mathbb{C}, \Gamma_0^-, \Gamma_1^-\big)$.

By $B_{\pm,0}$ we denote the self-adjoint extension of $B_\pm$ which is defined on

$$\mathrm{dom}(B_{\pm,0}) = \ker(\Gamma_0^\pm) \quad \text{by} \quad B_{\pm,0} = B_\pm^{\langle *\rangle}\big|_{\ker(\Gamma_0^\pm)}.$$

Clearly, $\big(\mathbb{C}, \Gamma_0^\pm, \Gamma_1^\pm\big)$ are also boundary triples for $A_\pm^{[*]}$. The Weyl functions $M_\pm(z)$ and the $\gamma$-fields $\gamma_{A_\pm}(z)$ of the operators $A_\pm$ corresponding to $\big(\mathbb{C}, \Gamma_0^\pm, \Gamma_1^\pm\big)$ are connected with the Weyl functions $m_\pm(z)$ and the $\gamma$-fields $\gamma_{B_\pm}(z)$ of the operators $B_\pm$ by the equalities

$$M_\pm(z) = m_\pm(\pm z), \quad \gamma_{A_\pm}(z) = \gamma_{B_\pm}(\pm z), \quad z \in \rho(B_{\pm,0}).$$

In the next lemma we reformulate the Veselić condition from Theorem 2.1 for the coupling $A$ of two nonnegative operators as defined in Definition 3.2, cf. [29] and [14].

**Lemma 3.6.** *Let conditions* (A1) *through* (A5) *be satisfied. Then the following statements hold:*

(i) *The coupling $A$ is definitizable and $\infty \in c(A)$.*

(ii) *$\infty \in c_r(A)$ if and only if there exists $\eta_0 > 0$ such that for all real $f_\pm \in \mathcal{H}_\pm$*

$$\int_{\eta_0}^{\eta} \mathrm{Re}\, \frac{\big(\widehat{f}_+(iy) + \widehat{f}_-(-iy)\big)^2}{m_+(iy) + m_-(-iy)} dy = O(1) \quad as \quad \eta \to +\infty. \tag{3.22}$$

(iii) $0 \notin c_s(A)$ *and* $\ker A = \ker A^2$ *if and only if there is* $\eta_0 > 0$ *such that for all real* $f_\pm \in \mathcal{H}_\pm$

$$\int_\eta^{\eta_0} \operatorname{Re} \frac{\left(\widehat{f}_+(iy) + \widehat{f}_-(-iy)\right)^2}{m_+(iy) + m_-(-iy)} dy = O(1) \quad as \quad \eta \downarrow 0. \tag{3.23}$$

*Proof.* (i) Since $m_+, m_- \in \mathcal{S}$, by Proposition 2.5 we have $\operatorname{Re}\left(m_+(iy) + m_-(-iy)\right) > 0$ for all $y \in \mathbb{R}_+$. Then, by Theorem 3.1(d) for the operator $A$ from Definition 3.2 we have $\rho(A) \neq \emptyset$.

The operator $A_0 = B_{+,0} \oplus (-B_{-,0})$ is definitizable, and $\infty \in c(A_0)$ as, by assumptions, $B_{\pm,0}$ are unbounded. Since the resolvent $(A - z)^{-1}$ of $A$ is a one-dimensional perturbation of the resolvent $(A_0 - z)^{-1}$, see (3.7), the claim (i) follows from [42, Theorem 1].

(ii) Applying Theorem 3.1 to the operators $A_+ = B_+$ and $A_- = -B_-$ in the inner product spaces $(\mathcal{K}_\pm, [\cdot, \cdot]_{\mathcal{K}_\pm}) = (\mathcal{H}_\pm, \pm\langle \cdot, \cdot \rangle_{\mathcal{H}_\pm})$ one obtains the equality

$$\left[(A - z)^{-1} f, f\right]_{\mathcal{K}} = \left[(A_0 - z)^{-1} f, f\right]_{\mathcal{K}} - \frac{[f, \gamma(\overline{z})]_{\mathcal{K}} [\gamma(z), f]_{\mathcal{K}}}{m_+(z) + m_+(-z)},$$

where $A_0 = B_{+,0} \oplus (-B_{-,0})$, $z \in \rho(A) \cap \rho(A_0)$. Since $A_0$ is a self-adjoint operator in $\mathcal{H}$ Theorem 2.1 and (3.12) imply that the condition $\infty \notin c_s(A)$ is equivalent to

$$\int_{\eta_0}^\eta \operatorname{Re} \frac{[f, \gamma(-iy)]_{\mathcal{K}} [\gamma(iy), f]_{\mathcal{K}}}{m_+(iy) + m_-(-iy)} dy = O(1) \quad \eta \to +\infty \quad \text{for all} \quad f \in \mathcal{K}. \tag{3.24}$$

Decompose $f \in \mathcal{K}$ into its "real" and "imaginary" part, $f = f^R + if^I$, where $f^R$ and $f^I$ are real, see (2.13). Since the vector valued functions $\gamma_{B_\pm}(z)$ are real, it follows from (2.12) and (2.14) that

$$\left[\gamma(iy), f^R\right]_{\mathcal{K}} = \left[f^R, \gamma(-iy)\right]_{\mathcal{K}}, \quad \left[\gamma(iy), f^I\right]_{\mathcal{K}} = \left[f^I, \gamma(-iy)\right]_{\mathcal{K}}$$

and hence

$$[f, \gamma(-iy)]_{\mathcal{K}} [\gamma(iy), f]_{\mathcal{K}} = \left[f^R + if^I, \gamma(-iy)\right]_{\mathcal{K}} [\gamma(iy), f^R + if^I]_{\mathcal{K}}$$
$$= \left[f^R, \gamma(-iy)\right]_{\mathcal{K}}^2 + \left[f^I, \gamma(-iy)\right]_{\mathcal{K}}^2.$$

Therefore, $\infty \notin c_s(A)$ if and only if (3.24) hold for all real $f \in \mathcal{K}$. Since

$$\left[f, \gamma(-iy)\right]_{\mathcal{K}} = \langle f_+, \gamma_{B_+}(-iy) \rangle_{\mathcal{H}_+} - \langle f_-, \gamma_{B_-}(iy) \rangle_{\mathcal{H}_-} = \widehat{f}_+(iy) - \widehat{f}_-(-iy), \tag{3.25}$$

condition (3.24) takes the form

$$\int_{\eta_0}^\eta \operatorname{Re} \frac{\left(\widehat{f}_+(iy) - \widehat{f}_-(-iy)\right)^2}{m_+(iy) + m_-(-iy)} dy = O(1) \quad \text{as} \quad \eta \to +\infty,$$

which reduces to (3.22) when we substitute $f_-$ with $-f_-$.

(iii) By Theorem 2.1 and (3.12) we have that the conjunction $0 \notin c_s(A)$ and $\ker A = \ker A^2$ is equivalent to

$$\int_\eta^{\eta_0} \operatorname{Re} \frac{[f, \gamma(-iy)]_{\mathcal{K}} [\gamma(iy), f]_{\mathcal{K}}}{m_+(iy) + m_-(-iy)} dy = O(1) \quad \text{as} \quad \eta \downarrow 0 \quad \text{for all} \quad f \in \mathcal{K}. \tag{3.26}$$

The reasoning in the proof of item (ii) shows that the preceding equivalence is preserved if $f$ in (3.26) is restricted to be real, which in view of (3.25) yields (3.23). $\qquad\square$

**Remark 3.7.** The coupling $A$ in Lemma 3.6 is not necessary non-negative, so it may have other critical points $\alpha$ distinct from 0. A criterion for $\alpha \notin c_s(A)$ can be derived from Lemma 3.6 (iii) by using the shift $A \to A - \alpha I$. However, we do not pursue this case here, since the main object of this paper, the operator $A$ in Section 5 associated with the indefinite Sturm-Liouville expression (1.1) is non-negative.

### 3.3. $D$-properties and conditions for regularity.

**Definition 3.8.** A pair of Nevanlinna functions $m_+$ and $m_-$ is said to have the $D_\infty$-*property* (resp. $D_0$-*property*) if

$$\frac{\max\{\operatorname{Im} m_+(iy), \operatorname{Im} m_-(iy)\}}{|m_+(iy) + m_-(-iy)|} = O(1) \quad \text{as} \quad y \to +\infty \quad (\text{resp. } y \downarrow 0). \qquad (3.27)$$

**Lemma 3.9.** *Assume that a pair $m_+$ and $m_-$ has the $D_\infty$-property (resp. $D_0$-property). Then*

$$\frac{m_+(iy) + m_-(iy)}{m_+(iy) + \overline{m_-(iy)}} = O(1) \quad as \quad y \to +\infty \quad (resp. \quad y \downarrow 0). \qquad (3.28)$$

*If, in addition, there exists $y_0 > 0$ such that*

$$\operatorname{Re} m_+(iy) \operatorname{Re} m_-(iy) > 0 \quad for\ all \quad y > y_0 \quad (resp. \quad 0 < y < y_0), \qquad (3.29)$$

*then*

$$\frac{m_+(iy) - \overline{m_-(iy)}}{m_+(iy) + \overline{m_-(iy)}} = O(1) \quad as \quad y \to +\infty \quad (resp. \quad y \downarrow 0). \qquad (3.30)$$

*Proof.* Assume that the pair $m_+$ and $m_-$ has $D_\infty$-property. To prove (3.28) we use the notation $u_\pm(iy)$ and $v_\pm(iy)$ introduced in (3.18). With this notation we have

$$\big|m_+(iy) + m_-(iy)\big|^2 = \big(u_+(iy) + u_-(iy)\big)^2 + \big(v_+(iy) + v_-(iy)\big)^2$$

and

$$\frac{|u_+(iy) + u_-(iy)|}{|m_+(iy) + \overline{m_-(iy)}|} < 1 \quad \text{for all} \quad y > 0.$$

By the $D_\infty$-property

$$\frac{v_+(iy) + v_-(iy)}{|m_+(iy) + \overline{m_-(iy)}|} = O(1) \quad \text{as} \quad y \to +\infty.$$

Hence (3.28) holds.

To prove (3.30), assume further that there exists $y_0 > 0$ such that (3.29) holds for all $y > y_0$. Then (3.29) yields

$$\frac{|\operatorname{Re} m_\pm(iy)|}{|m_+(iy) + \overline{m_-(iy)}|} < \frac{|\operatorname{Re} m_\pm(iy)|}{|\operatorname{Re} m_+(iy) + \operatorname{Re} m_-(iy)|} \le 1 \quad \text{for all} \quad y > y_0,$$

which, together with (3.27), imply (3.30).

To prove the claims involving the $D_0$-property we notice that if the pair $m_+(z)$ and $m_-(z)$ has the $D_0$-property, then the pair $m_+(-1/z)$ and $m_-(-1/z)$ has the $D_\infty$-property and we apply already proven statements to the functions $m_+(-1/z)$ and $m_-(-1/z)$. $\qquad \square$

As was shown in [51] and [49] the $D_\infty$-property ($D_0$-property, respectively) is necessary for the condition $\infty \notin c_s(A)$ ($0 \notin c_s(A)$, respectively).

In the next theorem we show that the $D_\infty$-property becomes also sufficient for $\infty \notin c_s(A)$ if it is supplemented by the assumption (3.29).

**Theorem 3.10.** *Let conditions* (A1) *through* (A5) *be satisfied and assume that there exists $y_0 > 0$ such that*

$$\operatorname{Re} m_+(iy) \operatorname{Re} m_-(iy) > 0 \quad \text{for all} \quad y > y_0. \tag{3.31}$$

*Then the coupling $A$ is definitizable in the Krein space $\big(\mathcal{K}, [\,\cdot\,,\cdot\,]_\mathcal{K}\big)$, $\infty \in c(A)$ and the following equivalence holds:*

$$\infty \in c_r(A) \quad \Leftrightarrow \quad \text{the pair } m_+ \text{ and } m_- \text{ has } D_\infty\text{-property.}$$

*Proof.* The definitizability of $A$ and $\infty \in c(A)$ follow from item (i) in Lemma 3.6.

The necessity of the condition that the pair $m_+$ and $m_-$ has $D_\infty$-property for $\infty \notin c_s(A)$ was proved in [51] and [49].

To prove sufficiency, assume that the pair $m_+$ and $m_-$ has $D_\infty$-property. We use Lemma 3.6(ii) to prove $\infty \in c_r(A)$. The integral in (3.22) can be rewritten as the sum $I_1(f_+, f_-) + I_2(f_+, f_-)$ of two integrals

$$I_1(f_+, f_-) = \int_{\eta_0}^{\eta} \frac{\operatorname{Re}\big((\widehat{f}_+(iy) + \widehat{f}_-(-iy))^2\big)\big(u_+(iy) + u_-(iy)\big)}{\left| m_+(iy) + \overline{m_-(iy)} \right|^2} dy,$$

$$I_2(f_+, f_-) = \int_{\eta_0}^{\eta} \frac{\operatorname{Im}\big((\widehat{f}_+(iy) + \widehat{f}_-(-iy))^2\big)\big(v_+(iy) - v_-(iy)\big)}{\left| m_+(iy) + \overline{m_-(iy)} \right|^2} dy,$$

where we use the notation introduced in (3.18).

We will prove that both of these integrals are bounded as $\eta \to +\infty$. By Lemma 3.9 there exist $y_1, C_1 > 0$ such that

$$\left| \frac{m_+(iy) + m_-(iy)}{m_+(iy) + \overline{m_-(iy)}} \right| \le C_1 \quad \text{for all} \quad y > y_1. \tag{3.32}$$

It follows from (3.32) and (3.15) that for all $\eta \ge \eta_0 > y_1$ we have

$$\int_{\eta_0}^{\eta} \left| \widehat{f}_\pm(iy) \right|^2 \frac{|u_+(iy) + u_-(iy)|}{\left| m_+(iy) + \overline{m_-(iy)} \right|^2} dy \le C_1^2 \int_{\eta_0}^{\eta} \left| \widehat{f}_\pm(iy) \right|^2 \frac{|u_+(iy) + u_-(iy)|}{|m_+(iy) + m_-(iy)|^2} dy$$

$$< 5\pi C_1^2 \|f_\pm\|^2.$$

This proves that

$$\left| I_1(f_+, f_-) \right| < 10\pi C_1^2 \big( \|f_+\|_{\mathcal{H}_+}^2 + \|f_-\|_{\mathcal{H}_-}^2 \big) \tag{3.33}$$

for all real $f_\pm \in \mathcal{H}_\pm$ and for all $\eta \ge \eta_0 > y_1$.

It follows from (3.30) that there exist $y_2, C_2 > 0$ such that

$$\left| \frac{m_+(iy) - \overline{m_-(iy)}}{m_+(iy) + \overline{m_-(iy)}} \right| \le C_2 \quad \text{for all} \quad y > y_2.$$

The preceding inequality yields that for all real $f_\pm \in \mathcal{H}_\pm$ and for all $\eta \geq \eta_0 > y_2$ we have

$$
\begin{aligned}
\left| I_2(f_+, f_-) \right| &\leq C_2^2 \int_{\eta_0}^{\eta} \frac{\left| \mathrm{Im}\big( (\widehat{f}_+(iy) + \widehat{f}_-(-iy))^2 \big) \right| |v_+(iy) - v_-(iy)|}{\left| m_+(iy) - \overline{m_-(iy)} \right|^2} dy \\
&\leq C_2^2 \int_{\eta_0}^{\eta} \frac{\left| \mathrm{Im}\big( (\widehat{f}_+(iy) + \widehat{f}_-(-iy))^2 \big) \right|}{\mathrm{Im}\big( m_+(iy) - \overline{m_-(iy)} \big)} dy,
\end{aligned} \tag{3.34}
$$

where for the second inequality we used that

$$
\left| m_+(iy) - \overline{m_-(iy)} \right| \geq v_+(iy) + v_-(iy) \geq |v_+(iy) - v_-(iy)| \quad \text{for all} \quad y > 0.
$$

Next we prove the inequality

$$
\int_{\eta_0}^{\eta} \frac{\left| \mathrm{Im}\big( (\widehat{f}_+(iy) + \widehat{f}_-(-iy))^2 \big) \right|}{\mathrm{Im}\big( m_+(iy) - \overline{m_-(iy)} \big)} dy \leq 2\pi \big( \|f_+\|_{\mathcal{H}_+}^2 + \|f_-\|_{\mathcal{H}_-}^2 \big) \tag{3.35}
$$

for all real $f_\pm \in \mathcal{H}_\pm$ and for all $\eta > \eta_0 > y_2$. This inequality will be deduced from (3.9) in Lemma 3.3 applied to the symmetric operator $S$ defined in Theorem 3.1 with the following specific setting.

The Krein space $\big( \mathcal{K}_+, [\cdot, \cdot]_{\mathcal{K}_+} \big)$ is the Hilbert spaces $\big( \mathcal{H}_+, \langle \cdot, \cdot \rangle_{\mathcal{H}_+} \big)$, the Krein space $\big( \mathcal{K}_-, [\cdot, \cdot]_{\mathcal{K}_-} \big)$ is the Hilbert spaces $\big( \mathcal{H}_-, \langle \cdot, \cdot \rangle_{\mathcal{H}_-} \big)$, the symmetric operator $A_+$ is given by $A_+ = B_+$ and the symmetric operator $A_-$ is given by $A_- = -B_-$. Notice that in this part of the proof the spaces $\mathcal{K}_+$ and $\mathcal{K}_-$ differ from those in (A1). The operator $A_\pm$ is a closed symmetric densely defined operator with defect numbers $(1,1)$ in the Hilbert space $\big( \mathcal{K}_\pm, [\cdot, \cdot]_{\mathcal{K}_\pm} \big)$. Furthermore, $\big( \mathbb{C}, \Gamma_0^+, \Gamma_1^+ \big)$ is a boundary triple for $A_+^{\langle * \rangle}$ (the adjoint in the Hilbert space $\mathcal{K}_+ = \mathcal{H}_+$) with the corresponding Weyl function and the $\gamma$−field given by

$$
z \mapsto m_+(z), \quad z \mapsto \gamma_{B_+}(z), \quad z \in \mathbb{C} \setminus \mathbb{R},
$$

while $\big( \mathbb{C}, \Gamma_0^-, -\Gamma_1^- \big)$ is a boundary triple for $A_-^{\langle * \rangle}$ (the adjoint in the Hilbert space $\mathcal{K}_- = \mathcal{H}_-$) with the corresponding Weyl function and the $\gamma$−field given by

$$
z \mapsto -m_-(-z), \quad z \mapsto \gamma_{B_-}(-z), \quad z \in \mathbb{C} \setminus \mathbb{R}.
$$

The operator $S$ defined in Theorem 3.1 on the domain (3.3) is a real densely defined symmetric operator with defect numbers $(1,1)$ acting in a Hilbert space $\mathcal{K}, [\cdot, \cdot]_{\mathcal{K}}$ which is the direct sum, $\mathcal{K} = \mathcal{H}_+ \oplus \mathcal{H}_-$, of the Hilbert spaces $\big( \mathcal{H}_+, \langle \cdot, \cdot \rangle_{\mathcal{H}_+} \big)$ and $\big( \mathcal{H}_-, \langle \cdot, \cdot \rangle_{\mathcal{H}_-} \big)$. A real boundary triple for $S^{\langle * \rangle}$ is the boundary triple $(\mathbb{C}, \Gamma_0, \Gamma_1)$ given in (3.5). The corresponding Weyl function $M(z)$ and the $\gamma$-field $\gamma(z)$ are given by

$$
M(z) = m_+(z) - m_-(-z), \quad \gamma(z) = \begin{pmatrix} \gamma_{B_+}(z) \\ \gamma_{B_-}(-z) \end{pmatrix}, \quad z \in \mathbb{C} \setminus \mathbb{R},
$$

see (3.6). For $f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in \mathcal{K} = \mathcal{H}_+ \oplus \mathcal{H}_-$, from (3.14), we obtain

$$
\widehat{f}(iy) = [f, \gamma(-iy)]_{\mathcal{K}} = \langle f_+, \gamma_{B_+}(-iy) \rangle_{\mathcal{H}_+} + \langle f_-, \gamma_{B_-}(iy) \rangle_{\mathcal{H}_-} = \widehat{f}_+(iy) + \widehat{f}_-(-iy).
$$

Now (3.35) follows by applying (3.9) in Lemma 3.3 to the real symmetric operator $S$ acting in the Hilbert space $\mathcal{K}$. Inequalities (3.34) and (3.35) yield

$$\left| I_2(f_+, f_-) \right| \leq C_2^2 2\pi \left( \|f_+\|_{\mathcal{H}_+}^2 + \|f_-\|_{\mathcal{H}_-}^2 \right) \tag{3.36}$$

for all real $f_\pm \in \mathcal{H}_\pm$ and for all $\eta > \eta_0 > y_2$. From (3.33) and (3.36) it follows that (3.22) holds. Hence Lemma 3.6(ii) implies $\infty \in c_r(A)$. $\qquad\square$

In the next theorem we give a criterion for $0 \notin c_s(A)$ formulated in terms of $D_0$-property.

**Theorem 3.11.** *Let conditions* (A1) *through* (A5) *be satisfied and assume that there exists $y_0 > 0$ such that the Weyl functions $m_+$ and $m_-$ satisfy the condition*

$$\operatorname{Re} m_+(iy) \operatorname{Re} m_-(iy) > 0 \quad \text{for all} \quad 0 < y < y_0.$$

*Then*

$$0 \notin c_s(A) \quad \text{and} \quad \ker A = \ker A^2 \quad \Leftrightarrow \quad \text{the pair } m_+ \text{ and } m_- \text{ has } D_0\text{-property.}$$

*Proof.* The necessity of the $D_0$-property for $0 \notin c_s(A)$ was proved in [51], [49].

To prove the sufficiency we will employ Lemma 3.6 and decompose the integral in (3.22) into a sum $I_1(f_+, f_-) + I_2(f_+, f_-)$ of two integrals

$$I_1(f_+, f_-) = \int_\eta^{\eta_0} \frac{\operatorname{Re}\left((\widehat{f}_+(iy) + \widehat{f}_-(-iy))^2\right)\left(u_+(iy) + u_-(iy)\right)}{\left| m_+(iy) + \overline{m_-(iy)} \right|^2} dy,$$

$$I_2(f_+, f_-) = \int_\eta^{\eta_0} \frac{\operatorname{Im}\left((\widehat{f}_+(iy) + \widehat{f}_-(-iy))^2\right)\left(v_+(iy) - v_-(iy)\right)}{\left| m_+(iy) + \overline{m_-(iy)} \right|^2} dy$$

The estimates for $I_1(f_+, f_-)$ and $I_2(f_+, f_-)$ for every $f_\pm \in \mathcal{H}_\pm$ similar to those in (3.33) and (3.36) follow in the same way as in the proof of Theorem 3.10. $\qquad\square$

**Remark 3.12.** The coupling $A$ in Lemma 3.6 is not necessarily non-negative, so it may have other critical points $\alpha$ distinct from 0. However, in the present paper we formulate the regularity conditions only for point 0 since the main object of Section 4, the operator $A$ associated with the indefinite Sturm-Liouville expression (1.1), is non-negative.

3.4. **One-sided sufficient conditions for regularity.** In the next theorem we give a one-sided condition which is sufficient for $\infty \notin c_s(A)$.

**Theorem 3.13.** *Let conditions* (A1) *through* (A5) *be satisfied and assume that:*
- (i) *there exists $y_0 > 0$ such that (3.31) holds for all $y > y_0$;*
- (ii) *either* $\operatorname{Im} m_+(iy) = O(\operatorname{Re} m_+(iy))$ *or* $\operatorname{Im} m_-(iy) = O(\operatorname{Re} m_-(iy))$ *as* $y \to +\infty$.

*Then the coupling $A$ of $A_+$ and $A_-$ is definitizable in the Krein space $\left(\mathcal{K}, [\,\cdot\,,\cdot\,]_\mathcal{K}\right)$, $\infty \in c(A)$ and*

$$\infty \in c_r(A).$$

*Proof.* The definitizability of $A$ and $\infty \in c(A)$ follow from item (i) in Lemma 3.6. Let us assume that $\operatorname{Im} m_+(iy) = O(\operatorname{Re} m_+(iy))$ as $y \to +\infty$ and show that the pair $m_+$, $m_-$ has the $D_\infty$-property. Indeed, in view of (i)

$$\left| \frac{\operatorname{Im} m_+(iy)}{m_+(iy) + \overline{m_-(iy)}} \right| \leq \frac{\operatorname{Im} m_+(iy)}{|\operatorname{Re} m_+(iy)|}$$

and by assumption (ii) there exist $C, y_0 > 0$, such that

$$\left| \frac{\operatorname{Im} m_+(iy)}{m_+(iy) + \overline{m_-(iy)}} \right| \leq C, \quad y > y_0. \tag{3.37}$$

Next, if $\operatorname{Im} m_-(iy) > 2 \operatorname{Im} m_+(iy)$, then

$$|\operatorname{Im} m_-(iy) - \operatorname{Im} m_+(iy)| \geq |\operatorname{Im} m_-(iy)| - |\operatorname{Im} m_+(iy)| > \frac{1}{2}|\operatorname{Im} m_-(iy)|$$

and hence

$$\left| \frac{\operatorname{Im} m_-(iy)}{m_+(iy) + \overline{m_-(iy)}} \right| \leq 2, \quad y > y_0.$$

Now, if $\operatorname{Im} m_-(iy) \leq 2 \operatorname{Im} m_+(iy)$, then

$$\left| \frac{\operatorname{Im} m_-(iy)}{m_+(iy) + \overline{m_-(iy)}} \right| \leq 2 \left| \frac{\operatorname{Im} m_+(iy)}{m_+(iy) + \overline{m_-(iy)}} \right| \leq 2C, \quad y > y_0.$$

Thus the pair $m_+$, $m_-$ has $D_\infty$-property, and the statement of Theorem 3.13 follows from Theorem 3.10. $\qquad\square$

In the next theorem we formulate a one-sided condition which is sufficient for $0 \notin c_s(A)$.

**Theorem 3.14.** *Let the conditions* (A1) *through* (A5) *be satisfied and assume that*
  (i) *there exist $y_0 > 0$ such that* (3.29) *holds;*
  (ii) *either* $\operatorname{Im} m_+(iy) = O\big(\operatorname{Re} m_+(iy)\big)$ *or* $\operatorname{Im} m_-(iy) = O\big(\operatorname{Re} m_-(iy)\big)$ *as $y \downarrow 0$.*
*Then $0 \notin c_s(A)$ and $\ker A = \ker A^2$.*

*Proof.* Let us assume that $\operatorname{Im} m_+(iy) = O(\operatorname{Re} m_+(iy))$ as $y \to 0$. Then in view of (i) and (ii) the inequality (3.37) holds for $0 < y < y_0$ and, hence, $\operatorname{Im} m_+(iy) = O\big(m_+(iy) + \overline{m_-(iy)}\big)$ as $y \downarrow 0$.

The proof of the relation $\operatorname{Im} m_-(iy) = O\big(m_+(iy) + \overline{m_-(iy)}\big)$ as $y \downarrow 0$ is similar to that in Theorem 3.13. Therefore, the pair $m_+$, $m_-$ has $D_0$-property, and the statement of Theorem 3.14 follows from Theorem 3.11. $\qquad\square$

## 4. Sturm-Liouville operator with indefinite weight

4.1. **Indefinite Sturm-Liouville operator as a coupling.** Let $I = (b_-, b_+)$ be a finite or infinite interval such that $-\infty \leq b_- < 0 < b_+ \leq +\infty$ and let $\mathfrak{a}$ be the differential expression (1.1) subject to the assumptions (1.2). In this section we study a nonnegative self-adjoint operator $A$ associated with $\mathfrak{a}$ in the Krein space $\big(L_w^2(I), [\cdot, \cdot]_w\big)$. In the definition of $A$ given in (4.13) we use nonnegative symmetric operators $B_\pm$ generated by the differential expressions $\mathfrak{b}_\pm$ in the Hilbert spaces $L_{w_\pm}^2(I_\pm)$ with the inner products

$$\langle f, g \rangle_{w_\pm} = \int_{I_\pm} f(x)\overline{g(x)}w_\pm(x)dx.$$

Let $B_{\pm,\max}$ be the maximal differential operator generated in $L_{w_\pm}^2(I_\pm)$ by the differential expression $\mathfrak{b}_\pm$ (see (1.6)), with the domain

$$\operatorname{dom}\big(B_{\pm,\max}\big) = \big\{ f \in L_{w_\pm}^2(I_\pm) : f, f^{[1]} \in AC_{\operatorname{loc}}(I_\pm), \ \mathfrak{b}_\pm(f) \in L_{w_\pm}^2(I_\pm) \big\},$$

where $f^{[1]}(x) := r(x)^{-1}f'(x), x \in I$. Let $B_{\pm,\min}(= (B_{\pm,\max})^{\langle*\rangle})$ be the minimal differential operator generated by $\mathfrak{b}_\pm$ in $L^2_{w_\pm}(I_\pm)$.

Let $z \in \mathbb{C} \setminus \mathbb{R}$ and denote by $s_\pm(\cdot, z)$ and $c_\pm(\cdot, z)$ the solutions on $I_\pm$ of the equation

$$\mathfrak{b}_\pm(f) = zf, \tag{4.1}$$

satisfying the boundary conditions

$$c_\pm(0, z) = 1, \ c_\pm^{[1]}(0, z) = 0 \quad \text{and} \quad s_\pm(0, z) = 0, \ s_\pm^{[1]}(0, z) = 1.$$

If $\mathfrak{b}_\pm$ is in the limit point case at $b_\pm$ then neither $s_\pm(\cdot, z)$ nor $c_\pm(\cdot, z)$ belongs to $L^2_{w_\pm}(I_\pm)$, however there exists a coefficient $m_\pm(z)$ such that the solution

$$\psi_\pm(t, z) = s_\pm(t, z) \mp m_\pm(z)c_\pm(t, z), \quad t \in I_\pm, \tag{4.2}$$

of the equation (4.1) belongs to $L^2_{w_\pm}(I_\pm)$.

In the limit point case the operator $B_\pm := B_{\pm,\min}$ is a symmetric operator in $L^2_{w_\pm}(I_\pm)$ with defect numbers $(1,1)$ and with the domain

$$\mathrm{dom}(B_\pm) = \left\{ f \in \mathrm{dom}(B_{\pm,\max}) : f(0) = f^{[1]}(0) = 0 \right\}. \tag{4.3}$$

In the limit circle case, by [46, Section 10.7], for every $f \in \mathrm{dom}(B_{\pm,\max})$ the following one-sided limit exists

$$f^{[1]}(b_\pm) := \lim_{x \to b_\pm \mp 0} r_\pm(x)^{-1}f'(x).$$

Let $m_\pm(z)$ be a coefficient such that the solution $\psi_\pm(x, z)$ in (4.2) satisfies the condition

$$\psi_\pm^{[1]}(b_\pm, z) = 0 \quad \text{for all} \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{4.4}$$

Clearly, $m_\pm(z)$ is calculated as $m_\pm(z) = \pm s^{[1]}(b_\pm, z)/c^{[1]}(b_\pm, z)$. In the limit circle case the operator $B_{\pm,\min}$ is a symmetric operator in $L^2_{w_\pm}(I_\pm)$ with defect numbers $(2,2)$ and we define its symmetric extension $B_\pm$ with defect numbers $(1,1)$ as the restriction of $\mathfrak{b}_\pm$ to the domain

$$\mathrm{dom}(B_\pm) = \left\{ f \in \mathrm{dom}(B_{\pm,\max}) : f(0) = f^{[1]}(0) = f^{[1]}(b_\pm) = 0 \right\}. \tag{4.5}$$

The adjoint operator $B_\pm^{\langle*\rangle}$ is the restriction of $\mathfrak{b}_\pm$ to the domain

$$\mathrm{dom}(B_\pm^{\langle*\rangle}) = \left\{ f \in \mathrm{dom}(B_{\pm,\max}) : f^{[1]}(b_\pm) = 0 \right\}.$$

In the following definition (see [58]) the notion of Neumann $m$-function is introduced both for the limit point case and the limit circle case.

**Definition 4.1.** The function $m_\pm(z)$ for which the solution $\psi_\pm(x, z)$ in (4.2) satisfies the condition

$$\left. \begin{array}{ll} \psi_\pm^{[1]}(b_\pm, z) = 0 & \text{if } \mathfrak{b}_\pm \text{ is in the limit circle case at } b_\pm \\ \psi_\pm(\cdot, z) \in L^2_{w_\pm}(I_\pm) & \text{if } \mathfrak{b}_\pm \text{ is in the limit point case at } b_\pm \end{array} \right\} \tag{4.6}$$

is called the *Neumann m-function of* $\mathfrak{b}_\pm$ *on* $I_\pm$ *subject to* (4.6).

The following proposition collects some facts from [27] about boundary triples for the operator $B_\pm^{\langle*\rangle}$.

**Proposition 4.2.** *Assume that* $\mathfrak{b}_\pm$ *satisfies* (1.2), *let* $B_\pm$ *be defined as in* (4.3) *or in* (4.5), *respectively, (depending on limit point or limit circle case) and let* $m_\pm$ *be the Neumann m-function of* $\mathfrak{b}_\pm$ *on* $I_\pm$, *subject to* (4.6). *Then:*

(a) $B_\pm$ is a symmetric nonnegative operator in the Hilbert space $L^2_{w_\pm}(I_\pm)$ with defect numbers $(1, 1)$.

(b) The triple $\left(\mathbb{C}, \Gamma_0^\pm, \Gamma_1^\pm\right)$, where

$$\Gamma_0^\pm f_\pm = f_\pm^{[1]}(0), \quad \Gamma_1^\pm f_\pm = \mp f_\pm(0), \quad f \in \mathrm{dom}(B_\pm^{\langle * \rangle}), \tag{4.7}$$

is a real boundary triple for $B_\pm^{\langle * \rangle}$.

(c) The Weyl function of $B_\pm$ corresponding to the boundary triple $\left(\mathbb{C}, \Gamma_0^\pm, \Gamma_1^\pm\right)$ coincides with the Neumann m-function $m_\pm(z)$. That is

$$m_\pm(z) = \mp \frac{\psi_\pm(0, z)}{\psi_\pm^{[1]}(0, z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{4.8}$$

If $\mathfrak{b}_\pm$ is in the limit circle case at $b_\pm$, then, in addition to (4.8), the following formula holds

$$m_\pm(z) = \pm \frac{s_\pm^{[1]}(b_\pm, z)}{c_\pm^{[1]}(b_\pm, z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{4.9}$$

(d) The Weyl function $m_\pm$ of $B_\pm$ belongs to the Stieltjes class $\mathcal{S}$ and satisfies the condition $\lim_{x \to -\infty} m_\pm(x) = 0$. In particular,

$$\mathrm{Re}\, m_\pm(iy) \geq 0 \quad \text{for all} \quad y > 0.$$

*Proof.* Since

$$\lim_{x \to b_\pm \mp 0} f^{[1]}(x) \overline{f(x)} = 0 \quad \text{for all} \quad f \in \mathrm{dom}(B_\pm^{\langle * \rangle})$$

both in the limit point case [47, Corollary, p. 199] and in the limit circle case [31, Lemma 2.1] the following formula holds

$$\int_{I_\pm} \mathfrak{b}_\pm(f_\pm) \overline{f_\pm} w_\pm dx = \pm f_\pm^{[1]}(0) \overline{f_\pm(0)} + \int_{I_\pm} \frac{1}{r_\pm} |f_\pm'|^2 dx, \quad f_\pm \in \mathrm{dom}(B_\pm^{\langle * \rangle}). \tag{4.10}$$

By (1.2) and Definition 2.3 this proves statements (a) and (b), see also [27, Proposition 9.51, Theorem 9.69].

The statement (c) is implied by Definition 2.4 and the equalities

$$\Gamma_0^\pm \psi_\pm(\cdot, z) = \psi_\pm^{[1]}(0, z) = 1, \quad \Gamma_1^\pm \psi_\pm(\cdot, z) = \mp \psi_\pm(0, z) = m_\pm(z) \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

The formula (4.9) follows from (4.4) and the equality

$$0 = \psi_\pm^{[1]}(b_\pm, z) = s_\pm^{[1]}(b_\pm, z) \mp m_\pm(z) c_\pm^{[1]}(b_\pm, z) \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

The extension $B_{\pm,0}$ of $B_\pm$ defined by

$$B_{\pm,0} f = B_\pm^{\langle * \rangle} f, \quad f \in \mathrm{dom}(B_{\pm,0}) := \ker \Gamma_0^\pm \tag{4.11}$$

is the von Neumann extension of $B_\pm$. Hence $B_{\pm,0} \geq 0$, see also (4.10), and thus the function $m_\pm$ is holomorphic on $\mathbb{R}_-$. Moreover, as it follows from [46, Theorem 3.1], see also [28, Proposition 3.6],

$$\lim_{x \to -\infty} m_\pm(x) = 0$$

and hence $m_\pm \in \mathcal{S}$. This proves (d).                                                 $\square$

With the differential expression $\mathfrak{a}$ we associate the following operator $A$ in the Krein $\left(L_w^2(I), [\,\cdot\,,\cdot\,]_w\right)$:

$$\mathrm{dom}(A) = \left\{ f \in \mathrm{dom}(B_+^{\langle *\rangle}) \oplus \mathrm{dom}(B_-^{\langle *\rangle}) : f, r^{-1}f' \in AC_{\mathrm{loc}}(I) \right\} \tag{4.12}$$

and

$$Af = \mathfrak{a}(f), \qquad f \in \mathrm{dom}(A). \tag{4.13}$$

**Lemma 4.3.** *For every $\lambda \in \mathbb{R}$ the subspace $\ker(A - \lambda I)$ is at most one-dimensional.*

*Proof.* Let $\lambda \in \mathbb{R}$. If $\mathfrak{b}_\pm$ is limit circle at $b_\pm$, then by Weyl's alternative the equation $\mathfrak{b}_\pm(f) = \pm\lambda f$ has two linearly independent solutions $c_\pm(x, \pm\lambda)$ and $s_\pm(x, \pm\lambda)$ in $L_{w_\pm}^2(I_\pm)$. Since the Wronskian of these solutions is not zero, it is not possible that both of these solutions satisfy $f^{[1]}(b_\pm) = 0$. Therefore, $\ker(B_\pm^{\langle *\rangle} \mp \lambda I)$ is one-dimensional.

If $\mathfrak{b}_\pm$ is limit point at $b_\pm$, then by Weyl's alternative the equation $\mathfrak{b}_\pm(f) = \pm\lambda f$ has at most one solution in $L_{w_\pm}^2(I_\pm)$. Consequently, $\ker(B_\pm^{\langle *\rangle} \mp \lambda I)$ is at most one-dimensional.

By the uniqueness theorem for linear initial value problems, the only solution of the problem $\mathfrak{b}_\pm(f) = \pm\lambda f$, $f_\pm(0) = f_\pm^{[1]}(0) = 0$ is the zero function. Therefore, the subspace

$$\ker(A - \lambda I) = \left\{ f = f_+ \oplus f_- : \begin{array}{ll} f_+ \in \ker(B_+^{\langle *\rangle} - \lambda I), & f_+(0) = f_-(0) \\ f_- \in \ker(B_-^{\langle *\rangle} + \lambda I), & f_+^{[1]}(0) = f_-^{[1]}(0) \end{array} \right\}$$

is also at most one-dimensional. $\qquad\qquad\square$

**Theorem 4.4.** *Let the differential expression $\mathfrak{b}$ satisfy* (1.2) *and let $m_\pm$ be the Neumann $m$-function of $\mathfrak{b}_\pm$ subject to* (4.6) *on $I_\pm$. Then the operator $A$ associated with the expression $\mathfrak{a}$ is the coupling of the operators $A_+ := B_+$ and $A_- := -B_-$ in the sense of Theorem* 3.1. *The operator $A$ is a nonnegative self-adjoint operator in the Krein space $\left(L_w^2(I), [\,\cdot\,,\cdot\,]_w\right)$ with $\rho(A) \neq \emptyset$ and $\infty \in c(A)$. We have*

(i) $\infty \in c_r(A) \quad\Leftrightarrow\quad$ *the pair $m_+$ and $m_-$ has $D_\infty$-property.*
(ii) $0 \notin c_s(A)$ *and* $\ker A = \ker A^2 \quad\Leftrightarrow\quad$ *the pair $m_+$ and $m_-$ has $D_0$-property.*
(iii) $\mathrm{Im}\, m_+(iy) = O\big(\mathrm{Re}\, m_+(iy)\big)$ *as $y \to +\infty$* $\;\Rightarrow\;$ $\infty \in c_r(A)$.
(iv) $\mathrm{Im}\, m_-(iy) = O\big(\mathrm{Re}\, m_-(iy)\big)$ *as $y \to +\infty$* $\;\Rightarrow\;$ $\infty \in c_r(A)$.
(v) $\mathrm{Im}\, m_+(iy) = O\big(\mathrm{Re}\, m_+(iy)\big)$ *as $y \downarrow 0$* $\;\Rightarrow\;$ $0 \notin c_s(A)$ *and* $\ker A = \ker A^2$.
(vi) $\mathrm{Im}\, m_-(iy) = O\big(\mathrm{Re}\, m_-(iy)\big)$ *as $y \downarrow 0$* $\;\Rightarrow\;$ $0 \notin c_s(A)$ *and* $\ker A = \ker A^2$.

*Proof.* The boundary triples $\left(\mathbb{C}, \Gamma_0^\pm, \Gamma_1^\pm\right)$ from Proposition 4.2 are also boundary triples for $A_\pm^+$. The coupling of the operators $A_\pm$ in Theorem 3.1 is characterized by the conditions

$$\Gamma_0^+(f_+) - \Gamma_0^-(f_-) = 0, \quad \Gamma_1^+(f_+) + \Gamma_1^-(f_-) = 0, \quad f_\pm \in \mathrm{dom}(B_\pm^{\langle *\rangle})$$

which in view of (4.7) can be rewritten as

$$f_+^{[1]}(0) = f_-^{[1]}(0), \quad f_+(0) = f_-(0), \quad f_\pm \in \mathrm{dom}(B_\pm^{\langle *\rangle}). \tag{4.14}$$

Therefore, the differential operator $A$ associated with the expression $\mathfrak{a}$ is the coupling of the operators $A_\pm := \pm B_\pm$ relative to the boundary triples $\left(\mathbb{C}, \Gamma_0^\pm, \Gamma_1^\pm\right)$. It

follows from (4.10) and (4.14) that for $f = f_+ + f_- \in \operatorname{dom} A$, $f_\pm \in \operatorname{dom}(B_\pm^{\langle * \rangle})$, we have

$$
\begin{aligned}
[Af, f]_w &= \left\langle B_+^{\langle * \rangle} f_+, f_+ \right\rangle_{w_+} + \left\langle B_-^{\langle * \rangle} f_-, f_- \right\rangle_{w_-} \\
&= f_+^{[1]}(0)\overline{f_+(0)} - f_-^{[1]}(0)\overline{f_-(0)} + \int_I \frac{1}{r}|f'|^2 dt = \int_I \frac{1}{r}|f'|^2 dt \geq 0.
\end{aligned}
$$

Hence the operator $A$ is nonnegative.

The Weyl function $M_\pm(z)$ of operator $A_\pm$ corresponding to $(\mathbb{C}, \Gamma_0^\pm, \Gamma_1^\pm)$ and the Weyl function $m_\pm(z)$ of operator $B_\pm$ satisfy

$$
M_\pm(z) = m_\pm(\pm z), \qquad z \in \mathbb{C} \setminus \mathbb{R}.
$$

Since $m_+, m_- \in \mathcal{S}$, by Proposition 2.5 we have $\operatorname{Re}\big(m_+(iy) + m_-(-iy)\big) > 0$ for all $y \in \mathbb{R}_+$. Consequently, Theorem 3.1 (d) yields $\rho(A) \neq \emptyset$. Therefore the operator $A$ is definitizable and $\infty \in c(A)$, see Lemma 3.6. As $m_+, m_- \in \mathcal{S}$, the assumptions of Theorems 3.10, 3.11, 3.13, and 3.14 are satisfied, and, thus, the remaining claims follow. $\qquad \square$

**Remark 4.5.** In the limit circle case Bennewitz, see [7], considered a more general class of Neumann $m$-functions than introduced in Definition 4.1. We restate Bennewitz's definition here.

Denote the Wronskian of two functions $f, g \in \operatorname{dom} B_{\pm,\max}$ by

$$
\mathsf{W}_t(f, g) := f(t)g^{[1]}(t) - f^{[1]}(t)g(t), \quad t \in I_\pm.
$$

The one-sided limit

$$
\mathsf{W}_{b_\pm}(f, g) := \lim_{x \to b_\pm \mp 0} \big(f(t)g^{[1]}(t) - f^{[1]}(t)g(t)\big)
$$

exists for all $f, g \in \operatorname{dom}(B_{\pm,\max})$. Furthermore, according to Titchmarsh [74] (see also [27, Theorem 9.69], every symmetric boundary condition at $b_\pm$ for arbitrary $f \in \operatorname{dom}(B_{\pm,\max})$ can be written as

$$
\mathsf{W}_{b_\pm}\big(f, (\cos \alpha)s_\pm(\cdot, z_0) + (\sin \alpha)c_\pm(\cdot, z_0)\big) = 0
$$

for some $\alpha \in (-\pi/2, \pi/2]$ and some $z_0 \in \mathbb{C} \setminus \mathbb{R}$.

If $m_\pm(z)$ is a coefficient for which the solution $\psi_\pm(t, z)$ in (4.2) satisfies the condition

$$
\mathsf{W}_{b_\pm}\big(\psi_\pm(\cdot, z), (\cos \alpha)s_\pm(\cdot, z_0) + (\sin \alpha)c_\pm(\cdot, z_0)\big) = 0, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{4.15}
$$

for some $\alpha \in (-\pi/2, \pi/2]$, then $m_\pm(z)$ is called the *Neumann m-function of* $\mathfrak{b}_\pm$ *on* $I_\pm$. Clearly, $m_\pm(z)$ can be expressed as

$$
m_\pm(z) = \frac{(\cos \alpha)\mathsf{W}_{b_\pm}\big(s_\pm(\cdot, z), s_\pm(\cdot, z_0)\big) + (\sin \alpha)\mathsf{W}_{b_\pm}\big(s_\pm(\cdot, z), c_\pm(\cdot, z_0)\big)}{(\cos \alpha)\mathsf{W}_{b_\pm}\big(c_\pm(\cdot, z), s_\pm(\cdot, z_0)\big) + (\sin \alpha)\mathsf{W}_{b_\pm}\big(c_\pm(\cdot, z), c_\pm(\cdot, z_0)\big)}, \quad z \in \mathbb{C} \setminus \mathbb{R}.
$$

Since all the symmetric boundary conditions at $b_\pm$ are included in the boundary condition (4.15), the boundary condition (4.4) is included as well. Therefore, the class of Neumann $m$-functions introduced in this remark contains the Neumann $m$-functions introduced in Definition 4.1.

4.2. **Asymptotic properties of $m$-functions.** V.A. Marčenko [60], I.S. Kac, M.G. Krein [44, 46] and Y. Kasahara [54] showed that the asymptotic behaviour of the Weyl function $m$ along the imaginary axes at $+\infty$ is closely related to the behaviour of the coefficients of the differential expression at 0. In this section we present some results in this direction from [6, 7] and their recent developments in [58].

Define the following four functions

$$W_\pm(x) := \int_0^x w_\pm(\xi)d\xi, \quad R_\pm(x) := \int_0^x r_\pm(\xi)d\xi, \quad x \in I_\pm. \qquad (4.16)$$

The functions $W_+$ and $R_+$ are positive and increasing on $I_+$, while $W_-$ and $R_-$ are negative and increasing functions on $I_-$. Define the function $F_\pm : R_\pm(I_\pm) \to \mathbb{R}_+$ as follows

$$F_\pm(x) := \frac{1}{xW_\pm\left(R_\pm^{-1}(x)\right)}, \quad x \in R_\pm(I_\pm). \qquad (4.17)$$

Here

$$R_-(I_-) = (c_-, 0), \quad R_+(I_+) = (0, c_+) \quad \text{with} \quad -\infty \le c_- < 0 < c_+ \le +\infty. \quad (4.18)$$

The function $F_+$ is decreasing and unbounded, and $F_-$ is an unbounded increasing function. Denote by $f_\pm$ the inverse of $F_\pm$. Notice that both $f_-$ and $f_+$ are defined in a neighbourhood of $+\infty$, the function $f_+$ is positive and decreasing, the function $f_-$ is negative and increasing, and

$$\lim_{x \to +\infty} f_-(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} f_+(x) = 0.$$

The following result was proved by F. Atkinson [3]; see also [7, Theorem 3.4] for the improved version which we use here. The concept of the Neumann $m$-function of $\mathfrak{b}_\pm$ on $I_\pm$ is used in the sense defined in Remark 4.5. For the concept of a slowly varying function at $0_\pm$ we refer to Definition A.1 in Appendix A.

**Theorem 4.6.** *Let $W_\pm$ and $R_\pm$ be the functions defined in* (4.16), *let $f_\pm$ be the inverse of the function defined in* (4.17) *and let $m_\pm$ be the Neumann $m$-function of $\mathfrak{b}_\pm$ on $I_\pm$. If $W_\pm \circ R_\pm^{-1}$ is a slowly varying function at $0_\pm$, then*

$$m_\pm(iy) \sim \pm i f_\pm(y) \quad as \quad y \to +\infty.$$

*Proof.* Assume that $W_\pm \circ R_\pm^{-1}$ is a slowly varying function at $0_\pm$. By Corollary A.8 this condition is equivalent to

$$\int_0^x R_\pm(\xi)\, dW_\pm(\xi) = o(R_\pm(x)W_\pm(x)) \quad \text{as} \quad |x| \downarrow 0 \quad \text{with} \quad x \in I_\pm. \qquad (4.19)$$

The claim about the function $m_+$ was proved in [7, Theorem 3.4]. We use this result to prove the claim about $m_-$. Let us set $\widehat{w}_+(x) = w_-(-x)$, $\widehat{r}_+(x) = r_-(-x)$, $x \in \widehat{I}_+ = (0, -b_-)$. Then the Hilbert space $L^2_{\widehat{w}_+}(\widehat{I}_+)$ consists of functions

$$\widehat{y}(x) := y(-x), \quad y \in L^2_{w_-}(I_-).$$

Let $\widehat{B}_+$ be the minimal operator generated in $L^2_{\widehat{w}_+}(\widehat{I}_+)$ by the differential expression

$$(\widehat{\mathfrak{b}}_+(\widehat{f}))(x) := -(\mathfrak{b}_-(f))(-x), \quad x \in \widehat{I}_+.$$

Then the Neumann $m$-function $\widehat{m}_+$ of $\widehat{\mathfrak{b}}_+$ on $\widehat{I}_+$ is connected with $m_-$ by

$$\widehat{m}_+(z) = -m_-(-z). \qquad (4.20)$$

Next the functions

$$\widehat{W}_+(x) := \int_0^x \widehat{w}_+(\xi)d\xi, \quad \widehat{R}_+(x) := \int_0^x \widehat{r}_+(\xi)d\xi, \quad x \in \widehat{I}_+$$

are connected with $W_-$ and $R_-$ by the equalities

$$\widehat{W}_+(x) = -W_-(-x), \quad \widehat{R}_+(x) = -R_-(-x), \quad x \in \widehat{I}_+ \tag{4.21}$$

and the inverse $\widehat{f}_+$ of $\widehat{F}_+ : x \in (0, \epsilon) \mapsto 1/(x\widehat{W}_+ \circ \widehat{R}_+^{-1}) \in \mathbb{R}_+$ is connected with the inverse $f_-$ of $F_- : x \in (-\epsilon, 0) \mapsto 1/(xW_- \circ R_-^{-1}(-x)) \in \mathbb{R}_+$ by the equality

$$\widehat{f}_+(y) = -f_-(y), \quad y \in \mathbb{R}_+. \tag{4.22}$$

It is easy to see, that $\widehat{W}_+$ and $\widehat{R}_+$ satisfy the condition (4.19). Therefore, by Theorem 4.6 $\widehat{m}_+(iy) \sim i\widehat{f}_+(y)$. Hence one obtains by (4.20), (4.21), (4.22)

$$m_-(iy) = -\overline{\widehat{m}_+(iy)} \sim -\overline{i\widehat{f}_+(y)} = i\widehat{f}_+(y) = -if_-(y). \qquad \square$$

The following lemma was proved by Bennewitz [6]. The condition that appears in [6] is equivalent to the definition of a positively increasing function, see Definition A.15 in Appendix A.

**Lemma 4.7.** *Let $m_\pm$ be the Neumann $m$-function of $\mathfrak{b}_\pm$ on $I_\pm$. Then*

$$\operatorname{Re} m_\pm(iy) = O(\operatorname{Im} m_\pm(iy)) \quad as \quad y \to \pm\infty \tag{4.23}$$

*if and only if the function $R_\pm \circ W_\pm^{-1}$ is positively increasing at $0_\pm$.*

Notice that the concept of the Neumann $m$-function of $\mathfrak{b}_\pm$ on $I_\pm$ in Lemma 4.7 is used in the sense defined in Remark 4.5, while in the rest of the paper we use Definition 4.1. Kostenko in [58, Corollary 2.7] proved the following analog of Lemma 4.7.

**Lemma 4.8.** *Let $m_\pm$ be the Neumann $m$-function of $\mathfrak{b}_\pm$ on $I_\pm$, subject to (4.6). Then*

$$\operatorname{Im} m_\pm(iy) = O(\operatorname{Re} m_\pm(iy)) \quad as \quad y \to \pm\infty \tag{4.24}$$

*if and only if the function $W_\pm \circ R_\pm^{-1}$ is positively increasing at $0_\pm$.*

Similar criteria for estimates (4.23) and (4.24) at 0 were proved by Kostenko in [58, Theorem 2.11 and Corollary 2.15].

**Lemma 4.9.** *Let $w_\pm, r_\pm \notin L^1(I_\pm)$ and let $m_\pm$ be the Neumann $m$-function of $\mathfrak{b}_\pm$ on $I_\pm$, subject to (4.6). Then*

$$\operatorname{Re} m_\pm(iy) = O(\operatorname{Im} m_\pm(iy)) \quad as \quad y \to 0_\pm$$

*if and only if the function $R_\pm \circ W_\pm^{-1}$ is positively increasing at $\pm\infty$.*

**Lemma 4.10.** *Let $w_\pm, r_\pm \notin L^1(I_\pm)$ and let $m_\pm$ be the Neumann $m$-function of $\mathfrak{b}_\pm$ on $I_\pm$, subject to (4.6). Then*

$$\operatorname{Im} m_\pm(iy) = O(\operatorname{Re} m_\pm(iy)) \quad as \quad y \to 0\pm$$

*if and only if the function $W_\pm \circ R_\pm^{-1}$ is positively increasing at $\pm\infty$.*

In the following lemma we consider the cases in which the conditions $w_\pm, r_\pm \notin L^1(I_\pm)$ are not satisfied.

**Lemma 4.11.** *Let $m_\pm$ be the Neumann $m$-function of $\mathfrak{b}_\pm$ on $I_\pm$, subject to (4.6).*

(i) *Let $a_\pm = \pm \lim_{x \to b_\pm} 1/W_\pm(x)$. Then $a_\pm \geq 0$ and the function $\widetilde{m}_\pm$ defined by*

$$m_\pm(z) = -\frac{a_\pm}{z} + \widetilde{m}_\pm(z), \quad z \in \mathbb{C}_+,$$

*belongs to $\mathcal{S}$ and*

$$\lim_{y \downarrow 0} y\widetilde{m}_\pm(iy) = 0.$$

*In particular, if $w_\pm \in L^1(I_\pm)$, then $a_\pm > 0$, $ym_\pm(iy) \sim ia_\pm$ at $0_+$ and*

$$\operatorname{Re} m_\pm(iy) = o(\operatorname{Im} m_\pm(iy)) \quad as \quad y \downarrow 0. \tag{4.25}$$

(ii) *If $r_\pm \in L^1(I_\pm)$ and $w_\pm \notin L^1(I_\pm)$, then*

$$\operatorname{Im} m_\pm(iy) = o(\operatorname{Re} m_\pm(iy)) \quad as \quad y \downarrow 0. \tag{4.26}$$

*Proof.* The claims (i) and (ii) appear in [58, Lemma 2.10]. For the proof of (i) see also [28, Propositions 3.6, 4.6].

We present a proof of (ii) here since its proof in [58] has a gap. Assume that $r_+ \in L^1(I_+)$, $w_+ \notin L^1(I_+)$ and consider the *dual differential expression*

$$\widehat{\mathfrak{b}}_+(f) := -\frac{1}{r_+(t)} \frac{d}{dt}\left(\frac{1}{w_+(t)} \frac{d}{dt} f\right) \quad \text{on} \quad I_+. \tag{4.27}$$

Since $w_+ \notin L^1(I_+)$ the differential expression $\widehat{\mathfrak{b}}_+$ is singular at $b_+$. By [58, Lemma 2.10] (see also [28, Theorem 5.2]) the Neumann $m$-function $\widehat{m}_+$ of $\widehat{\mathfrak{b}}_+$ on $I_+$, subject to (4.6), is connected with $m_+$ by the equality

$$\widehat{m}_+(z) = -\frac{1}{zm_+(z)}, \quad z \in \mathbb{C}_+. \tag{4.28}$$

Since $r_+ \in L^1(I_+)$, by (4.25) in (i) we have that $\widehat{m}_+$ satisfies the condition

$$\operatorname{Re} \widehat{m}_+(iy) = o(\operatorname{Im} \widehat{m}_+(iy)) \quad \text{as} \quad y \downarrow 0.$$

Now (4.28) yields

$$\lim_{y \downarrow 0} \frac{\operatorname{Im} m_+(iy)}{\operatorname{Re} m_+(iy)} = \lim_{y \downarrow 0} \frac{\operatorname{Re} \widehat{m}_+(iy)}{\operatorname{Im} \widehat{m}_+(iy)} = 0.$$

The proof of (ii) for $m_-$ is similar. $\qquad \square$

**Remark 4.12.** As was shown in [28, Theorem 5.2] the formula (4.28) holds if the differential expression $\mathfrak{b}_+$ is singular at $b_+$. If $\mathfrak{b}_+$ is regular at $b_+$ then (4.28) is not true anymore. Notice, that this fact was overlooked in [58].

4.3. **Regularity of the critical point $\infty$.** Statements (iii), (iv) of Theorem 4.4 can be restated as follows.

**Theorem 4.13.** *Let the differential expression $\mathfrak{b}_\pm$ satisfy (1.2) and let the functions $R_\pm$ and $W_\pm$ be defined by (4.16). If either $W_+ \circ R_+^{-1}$ is positively increasing at $0_+$ or $W_- \circ R_-^{-1}$ is positively increasing at $0_-$, then $\infty \in c_r(A)$.*

*Proof.* Let $m_\pm$ be the Neumann $m$-function of $\mathfrak{b}_\pm$ on $I_\pm$, subject to (4.6). By Proposition 4.2 (d) $m_+$ and $m_-$ belong to the Stieltjes class $\mathcal{S}$. Thus $m_+$ and $m_-$ satisfy the assumption (3.31) of Theorem 3.10. Assume that $W_+ \circ R_+^{-1}$ is positively increasing at $0_+$. Then by Lemma 4.8 condition (4.24) holds. Hence by Theorem 4.4 (iii) we have $\infty \in c_r(A)$. Similar argument proves the theorem if $W_- \circ R_-^{-1}$ is positively increasing at $0_-$. $\qquad \square$

**Example 4.14.** Let $I = (-1, 1)$. Consider differential operators $B_\pm$ generated by $\mathfrak{b}_\pm$ in $L^2(I_\pm)$, where $r_-$, $w_-$ are arbitrary subject to conditions (1.2) and $r_+ = 1$, and $w_+$ satisfies the condition:

$$w_+(x) = x^\alpha v_+(x), \quad x \in I_+, \quad \alpha > -1, \tag{4.29}$$

where $v_+(x)$ is slowly varying at $0_+$. Then by Karamata's characterization theorem, Theorem A.6, we have

$$W_+(x) = \int_0^x t^\alpha v_+(t) dt \sim \frac{x^{\alpha+1}}{\alpha+1} v_+(x) \quad \text{as} \quad x \downarrow 0,$$

and hence $W_+(x)$ is regularly varying at $0_+$ of order $\alpha + 1 > 0$ by Proposition A.2. Theorem 4.13 yields that $\infty \in c_r(A)$.

In the case when both $w_+$ and $w_-$ satisfy the condition (4.29) with $v_\pm \in C^1(\overline{I_\pm})$ and $\alpha > -1/2$ (so called Beals conditions) this result was obtained by R. Beals in [5], and by B. Ćurgus and H. Langer in [16] for $\alpha > -1$. That one-sided condition for the weight $w$ on $I_+$ is enough for $\infty \in c_r(A)$ was noticed by A. Fleige in [33].

**Lemma 4.15.** *Let $a \in \mathbb{R}_+$ and let $\alpha, \beta, f, g : [a, +\infty) \to \mathbb{C} \setminus \{0\}$ be functions such that $\alpha$ and $\beta$ are bounded,*

$$\lim_{x \to +\infty} \frac{\alpha(x)}{\beta(x)} = 1 \quad and \quad \lim_{x \to +\infty} \frac{f(x)}{g(x)} = 1. \tag{4.30}$$

*Then*

$$\frac{1}{\alpha(x) - f(x)} = O(1) \ as \ x \to +\infty \quad \Leftrightarrow \quad \frac{1}{\beta(x) - g(x)} = O(1) \ as \ x \to +\infty. \tag{4.31}$$

*Proof.* We will prove the equivalence of the negations of the statements in (4.31). The negation of the statement on the left-hand side of (4.31) is: There exists an increasing sequence $(x_n)$ in $[a, +\infty)$ such that

$$\lim_{n \to +\infty} x_n = +\infty \qquad \text{and} \qquad \lim_{n \to +\infty} \big(\alpha(x_n) - f(x_n)\big) = 0.$$

Since for all $n \in \mathbb{N}$ we have

$$\beta(x_n) - g(x_n) = \alpha(x_n) \left( \frac{\beta(x_n)}{\alpha(x_n)} - \frac{g(x_n)}{f(x_n)} \right) + \big(\alpha(x_n) - f(x_n)\big) \frac{g(x_n)}{f(x_n)}$$

and since $\alpha$ is bounded, (4.30) and the stated negation imply that the negation of the right-hand side of (4.31) holds. The proof of the converse is similar. $\qquad \square$

**Lemma 4.16.** *Let $a \in \mathbb{R}_+$ and let $f$ and $g$ be positive functions defined on $[a, +\infty)$. Then*

$$\left( \frac{f(x)}{g(x)} - 1 \right)^{-1} = O(1) \ as \ x \to +\infty$$

$$\Leftrightarrow \quad \left( \frac{g(x)}{f(x)} - 1 \right)^{-1} = O(1) \ as \ x \to +\infty. \tag{4.32}$$

*Proof.* The equivalence of the negations of the propositions in (4.32) is clear. $\quad \square$

Application of Theorem 4.4 (i) and Theorem 4.6 leads to the following characterization of regularity of critical point $\infty$ under the assumptions of Theorem 4.6.

**Theorem 4.17.** *Let the differential expression* $\mathfrak{a}$ *satisfy* (1.2). *Let* $W_\pm$ *and* $R_\pm$ *be the functions defined in* (4.16) *and assume that* $W_\pm \circ R_\pm^{-1}$ *is slowly varying function at* $0_\pm$. *Then the operator* $A$ *associated with* $\mathfrak{a}$ *is nonnegative in the Krein space* $\left(L_w^2(I), [\,\cdot\,, \cdot\,]_w\right)$, $\rho(A) \neq \emptyset$, $\infty$ *is a critical point of* $A$, *and*

$$\infty \in c_r(A) \quad \Leftrightarrow \quad \left(1 + \frac{W_-\left(R_-^{-1}(-x)\right)}{W_+\left(R_+^{-1}(x)\right)}\right)^{-1} = O(1) \quad as \quad x \downarrow 0.$$

*Proof.* Assume that $W_\pm \circ R_\pm^{-1}$ is slowly varying function at $0_\pm$. An immediate consequence of the definition in (4.17) is the equivalence

$$\left(1 + \frac{W_-\left(R_-^{-1}(-x)\right)}{W_+\left(R_+^{-1}(x)\right)}\right)^{-1} = O(1) \text{ as } x \downarrow 0 \quad \Leftrightarrow \quad \left(1 - \frac{F_+(x)}{F_-(-x)}\right)^{-1} = O(1) \text{ as } x \downarrow 0.$$

Recall that $F_+$ is unbounded decreasing, and $F_-$ is an unbounded increasing function. Since $W_\pm \circ R_\pm^{-1}$ is slowly varying at $0_\pm$, the function $F_\pm$ is regularly varying at $0_\pm$ with index $-1$, see the definition in (4.17). As the function $f_\pm$ is the inverse of $F_\pm$, Corollary A.13 yields the following equivalence

$$\left(1 - \frac{F_+(x)}{F_-(-x)}\right)^{-1} = O(1) \text{ as } x \downarrow 0 \quad \Leftrightarrow \quad \left(1 + \frac{f_+(y)}{f_-(y)}\right)^{-1} = O(1) \text{ as } y \to +\infty.$$

Let $m_\pm$ be the Neumann $m$-function of $\mathfrak{b}_\pm$ on $I_\pm$. By Theorem 4.6 we have

$$\mp i\, m_\pm(iy) \sim f_\pm(y) \quad \text{as} \quad y \to +\infty.$$

The preceding asymptotic relation and Lemma 4.15 imply

$$\frac{\operatorname{Im} m_+(iy)}{m_+(iy) + m_-(-iy)} = O(1) \text{ as } y \to +\infty$$

$$\Leftrightarrow \quad \left(1 + \frac{f_-(y)}{f_+(y)}\right)^{-1} = O(1) \text{ as } y \to +\infty.$$

To see how Lemma 4.15 applies here we write

$$\frac{i\, \operatorname{Im} m_+(iy)}{m_+(iy) + m_-(-iy)} = \frac{1}{\frac{-i\, m_+(iy)}{\operatorname{Im} m_+(iy)} - \frac{i\, m_-(-iy)}{\operatorname{Im} m_+(iy)}},$$

set

$$\alpha(y) = \frac{-i\, m_+(iy)}{\operatorname{Im} m_+(iy)}, \quad f(y) = \frac{i\, m_-(-iy)}{\operatorname{Im} m_+(iy)}, \quad \beta(y) = 1, \quad g(y) = -\frac{f_-(y)}{f_+(y)},$$

and observe that the above asymptotic relation from Theorem 4.6 implies

$$\lim_{y \to +\infty} \alpha(y) = 1 \quad \text{and} \quad \lim_{y \to +\infty} \frac{f(y)}{g(y)} = 1.$$

Since by Lemma 4.16 we have

$$\left(1 + \frac{f_-(y)}{f_+(y)}\right)^{-1} = O(1) \text{ as } y \to +\infty \quad \Leftrightarrow \quad \left(1 + \frac{f_+(y)}{f_-(y)}\right)^{-1} = O(1) \text{ as } y \to +\infty,$$

we have proved that

$$\left(1 + \frac{W_-\left(R_-^{-1}(-x)\right)}{W_+\left(R_+^{-1}(x)\right)}\right)^{-1} = O(1) \text{ as } x \downarrow 0$$

$$\Leftrightarrow \quad \frac{\operatorname{Im} m_+(iy)}{m_+(iy) + m_-(-iy)} = O(1) \text{ as } y \to +\infty.$$

Similarly, we can prove that

$$\left(1 + \frac{W_-\left(R_-^{-1}(-x)\right)}{W_+\left(R_+^{-1}(x)\right)}\right)^{-1} = O(1) \text{ as } x \downarrow 0$$

$$\Leftrightarrow \quad \frac{\operatorname{Im} m_-(iy)}{m_+(iy) + m_-(-iy)} = O(1) \text{ as } y \to +\infty.$$

Therefore,

$$\left(1 + \frac{W_-\left(R_-^{-1}(-x)\right)}{W_+\left(R_+^{-1}(x)\right)}\right)^{-1} = O(1) \text{ as } x \downarrow 0$$

$$\Leftrightarrow \quad \text{the pair } m_+ \text{ and } m_- \text{ has } D_\infty\text{-property.}$$

Now the theorem follows from Theorem 4.4. $\qquad\qquad\square$

**Remark 4.18.** The criteria in Theorem 4.17 nicely complements the result of Kostenko in [58, Corollary 4.8(i)]. To see this, we notice that [58, Corollary 4.8(i)] can be restated as follows: If $W_- \circ R_-^{-1}$ is slowly varying function at $0_-$, $W_+ \circ R_+^{-1}$ is slowly varying function at $0_+$ and $\infty \in c_r(A)$, then $w$ is not odd or $r$ is not even. The "only if" part of Theorem 4.17 gives more than the fact that $w$ is not odd or $r$ is not even, that is, there exists a small enough $x$ such that $W_+\left(R_+^{-1}(x)\right) \neq -W_-\left(R_-^{-1}(-x)\right)$. The "only if" part of Theorem 4.17 gives

$$\left(1 + \frac{W_-\left(R_-^{-1}(-x)\right)}{W_+\left(R_+^{-1}(x)\right)}\right)^{-1} = O(1) \quad \text{as} \quad x \downarrow 0. \tag{4.33}$$

In this setting the negation of (4.33), that is the condition

$$W_+\left(R_+^{-1}(x)\right) \overset{\varsigma}{\sim} -W_-\left(R_-^{-1}(-x)\right) \quad \text{at} \quad 0_+,$$

appears to be a natural generalization of the condition that the functions $|w|$ and $r$ are even. Here we use the symbol $\overset{\varsigma}{\sim}$ which is defined in Definition A.9. In the case when $r = 1$, this condition also generalizes the condition of $w$ being odd-dominated which was used in Fleige's criterion for $\infty \in c_r(A)$, see [15, Definition 3.8 and Theorem 3.11].

For slowly varying functions the following corollary extends the result of [15, Corollary 3.15].

**Corollary 4.19.** *Let* $0 < b_+ \leq +\infty$, $I_+ = [0, b_+)$ *and* $r_+, w_+ \in L^1_{\text{loc}}(I_+)$ *be positive functions. Let* $\alpha, \beta \in \mathbb{R}_+$, *set* $b_- = -b_+/\beta$ *and define*

$$r(x) = \begin{cases} r_+(x) & \text{if } x \in [0, b_+) \\ \alpha r_+(-\beta x) & \text{if } x \in (b_-, 0), \end{cases} \quad w(x) = \begin{cases} w_+(x) & \text{if } x \in [0, b_+) \\ -\alpha w_+(-\beta x) & \text{if } x \in (b_-, 0). \end{cases}$$

*Let* $W_+$ *and* $R_+$ *be the functions defined in* (4.16) *and assume that* $W_+ \circ R_+^{-1}$ *is slowly varying function at* $0_+$. *Then the operator* $A$ *associated with* $\mathfrak{a}$ *is nonnegative in the Krein space* $\left(L^2_w(I), [\cdot, \cdot]_w\right)$, $\rho(A) \neq \emptyset$, $\infty$ *is a critical point of* $A$, *and* $\infty \in c_r(A)$ *if and only if* $\alpha \neq \beta$.

*Proof.* To apply Theorem 4.17 we first calculate for $x \in (c_-, 0)$ (cf. (4.18))

$$W_-\big(R_-^{-1}(x)\big) = -(\alpha/\beta)W_+\big(R_+^{-1}\big(-(\beta/\alpha)x\big)\big).$$

Hence $W_- \circ R_-^{-1}$ is a slowly varying function at $0_-$. Further

$$\frac{W_-\big(R_-^{-1}(-x)\big)}{W_+\big(R_+^{-1}(x)\big)} = -\frac{\alpha}{\beta}\frac{W_+\big(R_+^{-1}\big((\beta/\alpha)x\big)\big)}{W_+\big(R_+^{-1}(x)\big)},$$

and since $W_+ \circ R_+^{-1}$ is a slowly varying function at $0_+$ we have

$$\lim_{x \downarrow 0} \frac{W_-\big(R_-^{-1}(-x)\big)}{W_+\big(R_+^{-1}(x)\big)} = -\frac{\alpha}{\beta}.$$

Therefore

$$\left(1 + \frac{W_-\big(R_-^{-1}(-x)\big)}{W_+\big(R_+^{-1}(x)\big)}\right)^{-1} = O(1) \quad \text{as} \quad x \downarrow 0$$

holds if and only if $\alpha \neq \beta$. Now the claim follows from Theorem 4.17. $\qquad\square$

We illustrate Corollary 4.19 with an example which has appeared in [15, Example 3.17]. The novelty here is that we can give a characterization of the regularity of the critical point $\infty$ for all positive coefficients $\alpha$ and $\beta$.

**Example 4.20.** Let $w_+, r_+ : (0, 1) \to \mathbb{R}_+$ be given by

$$w_+(x) = \frac{1}{x(\ln x)^2}, \quad r_+(x) = 1, \quad x \in (0, 1).$$

Then

$$W_+(x) = W_+\big(R_+^{-1}(x)\big) = -\frac{1}{\ln x}, \quad x \in [0, 1).$$

Hence, $W_+ \circ R_+^{-1}$ is a slowly varying function at $0_+$. Therefore the operator $A$ from Corollary 4.19 is nonnegative in the Krein space $\big(L_w^2(-1, 1), [\,\cdot\,,\cdot\,]_w\big)$, $\rho(A) \neq \emptyset$, $\infty$ is its critical point and $\infty \in c_r(A)$ if and only if $\alpha \neq \beta$.

**Example 4.21.** Let $\alpha_\pm > 0$ and $I = (-1, 1)$. Let $r = 1$ on $I$ and

$$w_-(x) = \frac{\alpha_-}{x\big(-\ln(-x)\big)^{1+\alpha_-}}, \quad x \in (-1, 0), \quad w_+(x) = \frac{\alpha_+}{x(-\ln x)^{1+\alpha_+}}, \quad x \in (0, 1).$$

Then

$$R_-(x) = x, \quad x \in [-1, 0], \quad R_+(x) = x, \quad x \in [0, 1], \tag{4.34}$$

$$W_-(x) = \frac{-1}{\big(-\ln(-x)\big)^{\alpha_-}}, \quad x \in (-1, 0], \quad W_+(x) = \frac{1}{\big(-\ln x\big)^{\alpha_+}}, \quad x \in [0, 1). \tag{4.35}$$

Thus $W_- \circ R_-^{-1} = W_-$ is slowly varying at $0_-$, $W_+ \circ R_+^{-1} = W_+$ is slowly varying at $0_+$ and

$$\left(1 + \frac{W_-(-x)}{W_+(x)}\right)^{-1} = \left(1 - \big(-\ln(x)\big)^{\alpha_+ - \alpha_-}\right)^{-1} = O(1) \quad \text{as} \quad x \downarrow 0$$

holds if and only if $\alpha_+ \neq \alpha_-$.

By Theorem 4.17 the operator $A$ associated with the differential expression $\mathfrak{a}$ with the above defined $w$ and $r$ is nonnegative in the Krein space $\big(L_w^2(I), [\,\cdot\,,\cdot\,]_w\big)$, $\rho(A) \neq \emptyset$, $\infty$ is its critical point and $\infty \in c_r(A)$ if and only if $\alpha_+ \neq \alpha_-$. That is, $\infty$ is a singular critical point of $A$ if and only if $\alpha_+ = \alpha_-$. Notice that the implication

$$\alpha_+ = \alpha_- \quad \Rightarrow \quad \infty \in c_s(A)$$

follows from a result of Parfenov [61, Theorem 6], as with $\alpha_+ = \alpha_-$ the weight function $w(x), x \in I$, is odd on $I$.

The converse of the last displayed implication does not follow from neither of the following sufficient conditions for regularity: Volkmer's condition, see [76, Corollary 2.7] or [15, Theorem 3.14], Fleige's condition for odd-dominated weights, see [15], Parfenov's condition [62, Corollary 8] for non-odd weights.

4.4. **Discreteness.** For a closed operator $T$ its discrete spectrum consists of its isolated eigenvalues of finite algebraic multiplicity. The complement of the discrete spectrum is called the essential spectrum of $T$; it is denoted by $\sigma_{\mathrm{ess}}(T)$. The following theorem is based on a result from [43].

**Theorem 4.22.** *Let $0 < b_+ \le +\infty$, let $w_+$ and $r_+$ be almost everywhere positive functions on $(0, b_+)$ such that $w_+ \in L^1(0, b_+)$ and $r_+ \notin L^1(0, b_+)$ and let $B_+$ and $B_{+,0}$ be defined as in (4.5) and (4.11). Then the following two equivalences hold:*

$$\sigma_{\mathrm{ess}}(B_{+,0}) = \emptyset \quad \Leftrightarrow \quad \lim_{x \to b_+} R_+(x) \int_x^{b_+} w_+(\xi) d\xi = 0, \qquad (4.36)$$

$$0 \notin \sigma_{\mathrm{ess}}(B_{+,0}) \quad \Leftrightarrow \quad \sup_{x \in (0, b_+)} R_+(x) \int_x^{b_+} w_+(\xi) d\xi < +\infty \qquad (4.37)$$

*and in the latter case $0 \in \widehat{\rho}(B_+)$.*

*Proof.* Let us introduce the change of variable function by $\xi = R_+(x)$, $x \in (0, b_+)$ and consider the weight function $\widetilde{w}$ and the differential expression $\widetilde{\mathfrak{b}}$ by

$$\widetilde{w}(\xi) = \frac{w\big(R_+^{-1}(\xi)\big)}{r\big(R_+^{-1}(\xi)\big)}, \quad \widetilde{\mathfrak{b}} = -\frac{1}{\widetilde{w}} \frac{d^2}{dx^2} \quad \xi \in \mathbb{R}_+.$$

Let $\widetilde{B}$ and $\widetilde{B}_0$ be operators associated with $\widetilde{\mathfrak{b}}$ via (4.5) and (4.11). A straightforward verification yields that the operator $U$ defined for an $f : (0, b_+) \to \mathbb{C}$ by

$$(Uf)(\xi) = f\big(R_+^{-1}(\xi)\big), \quad \xi \in \mathbb{R}_+,$$

is a unitary operator from $L^2_{w_+}(0, b)$ to $L^2_{\widetilde{w}}(\mathbb{R}_+)$ and $\widetilde{B}_0 = U B_{+,0} U^{-1}$. That is the operators $B_{+,0}$ and $\widetilde{B}_0$ are unitarily equivalent.

We will use a result from [43]. It was shown in [43] that under the assumption $\widetilde{w} \in L^1(\mathbb{R}_+)$ the following equivalences hold:

$$\sigma_{\mathrm{ess}}(\widetilde{B}_0) = \emptyset \quad \Leftrightarrow \quad \lim_{\xi \to +\infty} \xi \int_\xi^{+\infty} \widetilde{w}(\tau) d\tau = 0, \qquad (4.38)$$

$$0 \notin \sigma_{\mathrm{ess}}(\widetilde{B}_0) \quad \Leftrightarrow \quad \sup_{\xi \in \mathbb{R}_+} \xi \int_\xi^{+\infty} \widetilde{w}(\tau) d\tau < +\infty. \qquad (4.39)$$

Moreover, in the latter case $0$ is a regular point for the Dirichlet extension of $\widetilde{B}$ and hence $0 \in \widehat{\rho}(\widetilde{B})$. The equivalences in the lemma follow from (4.38) and (4.39) since by the change of variable $\tau = R_+(t)$, $t \in (0, b_+)$, we have $\xi = R_+(x)$ and

$$\xi \int_\xi^{+\infty} \widetilde{w}(\tau) d\tau = R_+(x) \int_x^{b_+} w_+(t) dt, \quad \xi \in \mathbb{R}_+.$$

$\square$

Let $\mathfrak{b}_+$ be the dual differential expression (4.27). The spectrum of the Neumann realization $B_{+,0}$ (resp. $\widehat{B}_{+,0}$) of $\mathfrak{b}_+$ (resp. $\widehat{\mathfrak{b}}_+$) coincides with the support of the measure $d\sigma_+$ (resp. $d\widehat{\sigma}_+$) in the integral representation (2.10) of $m_+$ (resp. $\widehat{m}_+$). Since the functions $m_+$ and $\widehat{m}_+$ are related by (4.28) the essential spectra of the operators $B_{+,0}$ and $\widehat{B}_{+,0}$ coincide, see also [46, (12.5)] for the case $r_+ = 1$. Therefore, by switching the coefficients $w_+$ and $r_+$ we obtain the following statement.

**Corollary 4.23.** *Let $0 < b_+ \leq +\infty$ and let $w_+$ and $r_+$ be almost everywhere positive functions on $(0, b_+)$ such that $w_+ \notin L^1(0, b_+)$ and $r_+ \in L^1(0, b_+)$. The following two equivalences hold:*

$$\sigma_{\mathrm{ess}}(B_{+,0}) = \emptyset \quad \Leftrightarrow \quad \lim_{x \to b_+} W_+(x) \int_x^{b_+} r_+(\xi)d\xi = 0, \tag{4.40}$$

$$0 \notin \sigma_{\mathrm{ess}}(B_{+,0}) \quad \Leftrightarrow \quad \sup_{x \in (0,b_+)} W_+(x) \int_x^{b_+} r_+(\xi)d\xi < +\infty. \tag{4.41}$$

**Remark 4.24.** In the case $r_+ = 1$ conditions (4.36), (4.37) were proved in [43]. Conditions similar to (4.36) can be found also in [21] for Sturm-Liouville operators and in [69], [67] for canonical systems. Clearly, the equivalences (4.36), (4.37), (4.40) and (4.41) remain in force for $B_{-,0}$ with $b_+$, $w_+$, $r_+$ replaced by $b_-$, $w_-$, $r_-$, respectively. In particular,

$$\sigma_{\mathrm{ess}}(B_{-,0}) = \emptyset \quad \Leftrightarrow \quad \lim_{x \to b_-} R_-(x) \int_x^{b_-} w_-(\xi)d\xi = 0,$$

$$0 \notin \sigma_{\mathrm{ess}}(B_{-,0}) \quad \Leftrightarrow \quad \sup_{x \in (0,b_-)} R_-(x) \int_x^{b_-} w_-(\xi)d\xi < +\infty. \tag{4.42}$$

The next proposition shows that the spectrum of the operator $B_{\pm,0}$ defined in (4.11) can be discrete even in the limit point case.

**Proposition 4.25.** *Assume $w_\pm \in L^1(I_\pm)$. Then*

$$\int_{I_\pm} |R_\pm(\xi)| w_\pm(\xi)d\xi = \int_{I_\pm} |W_\pm(b_\pm) - W_\pm(\xi)| r_\pm(\xi)d\xi, \tag{4.43}$$

*meaning that either the two integrals diverge simultaneously, or, if one converges, then the other one converges as well and the integrals are equal. Further, if $R_\pm \in L^1_{w_\pm}(I_\pm)$, then the spectrum of $B_{\pm,0}$ is discrete and $0 \in \widehat{\rho}(B_\pm)$.*

*Proof.* Assume that $w_\pm \in L^1(I_\pm)$. That is assume that $W_\pm(b_\pm)$ is a real number. Further assume that the integral

$$\int_{I_\pm} |R_\pm(\xi)| w_\pm(\xi)d\xi = \int_0^{b_\pm} R_\pm(\xi)dW_\pm(\xi) = \lim_{x \to b_\pm} \int_0^x R_\pm(\xi)dW_\pm(\xi) \tag{4.44}$$

converges. Applying integration by parts (see, e.g., [77, Theorem 2.21]) for all $x \in I_\pm$ we obtain,

$$\int_0^x \big(W_\pm(x) - W_\pm(\xi)\big)dR_\pm(\xi) = \int_0^x R_\pm(\xi)dW_\pm(\xi). \tag{4.45}$$

Since the limit as $x \to b_\pm$ of the integral on the right-hand side of (4.45) exists, the Lebesgue Monotone Convergence Theorem implies that the limit of the integral on

the left-hand side of (4.45) exists and

$$\lim_{x \to b_\pm} \int_0^x \big(W_\pm(x) - W_\pm(\xi)\big) dR_\pm(\xi) = \int_0^{b_\pm} \big(W_\pm(b_\pm) - W_\pm(\xi)\big) dR_\pm(\xi). \quad (4.46)$$

Clearly,

$$\int_{I_\pm} \big|W_\pm(b_\pm) - W_\pm(\xi)\big| r_\pm(\xi) d\xi = \int_0^{b_\pm} \big(W_\pm(b_\pm) - W_\pm(\xi)\big) dR_\pm(\xi)$$

and the equality in (4.43) follows from (4.44), (4.45) and (4.46).

Assume now that the integral on the right-hand side of (4.43) converges. Then (4.46) holds. Applying again integration by parts for all $x \in I_\pm$ we obtain

$$\int_0^x \big(W_\pm(b_\pm) - W_\pm(\xi)\big) dR_\pm(\xi)$$
$$= R_\pm(x)\big(W_\pm(b_\pm) - W_\pm(x)\big) + \int_0^x R_\pm(\xi) dW_\pm(\xi). \quad (4.47)$$

Notice that all three terms in (4.47) are positive and that the integrals in (4.47) are increasing functions of $x \in I_\pm$. Therefore (4.46) implies that (4.44) holds. That is the first integral in (4.43) converges. Now the equality of the integrals follows from the first part of this proof.

Furthermore, taking the limit as $x \to b_\pm$ in (4.47) yields

$$\lim_{x \to b_\pm} R_\pm(x)\big(W_\pm(b_\pm) - W_\pm(x)\big) = 0. \quad (4.48)$$

This limit is the right-hand side of the equivalence (4.37) in Theorem 4.22. Hence the spectrum of $B_{\pm,0}$ is discrete. $\qquad \square$

The next theorem combines the results of Theorem 4.22, Corollary 4.23 and Theorem 4.17 to provide a necessary and sufficient condition for the existence of a Riesz basis consisting of eigenfunctions of the differential operator $A$.

**Theorem 4.26.** *Let the differential expression* $\mathfrak{a}$ *satisfy* (1.2) *and let* $W_\pm$ *and* $R_\pm$ *be the functions defined in* (4.16). *Assume*

(a) *The functions $w_+$ and $r_+$ satisfy one of the following three conditions:*
  (i) $w_+ \in L^1(I_+)$ *and* $r_+ \in L^1(I_+)$.
  (ii) $w_+ \in L^1(I_+)$, $r_+ \notin L^1(I_+)$ *and* $\lim_{x \uparrow b_+} R_+(x)\big(W_+(b_+) - W_+(x)\big) = 0$.
  (iii) $w_+ \notin L^1(I_+)$, $r_+ \in L^1(I_+)$ *and* $\lim_{x \uparrow b_+} W_+(x)\big(R_+(b_+) - R_+(x)\big) = 0$.
(b) *The functions $w_-$ and $r_-$ satisfy one of the following three conditions*
  (i) $w_- \in L^1(I_-)$ *and* $r_- \in L^1(I_-)$.
  (ii) $w_- \in L^1(I_-)$, $r_- \notin L^1(I_-)$ *and* $\lim_{x \downarrow b_-} R_-(x)\big(W_-(b_-) - W_-(x)\big) = 0$.
  (iii) $w_- \notin L^1(I_-)$, $r_- \in L^1(I_-)$ *and* $\lim_{x \uparrow b_-} W_-(x)\big(R_-(b_-) - R_-(x)\big) = 0$.

*Then the spectrum of the operator $A$ associated with the differential expression* $\mathfrak{a}$ *in the Hilbert space* $L^2_{|w|}(I)$ *is real and discrete, its eigenvalues accumulate on both sides of* $\infty$, *all nonzero eigenvalues are simple and Jordan chain at $0$ is of length at most $2$. The following statements hold.*

(A) *If either $W_+ \circ R_+^{-1}$ is positively increasing at $0_+$ or $W_- \circ R_-^{-1}$ is positively increasing at $0_-$, then we have:*
  (I) *There exists a Riesz basis of the Hilbert space* $L^2_{|w|}(I)$ *which consists of generalized eigenfunctions of $A$.*

(B) *If $W_+ \circ R_+^{-1}$ is slowly varying at $0_+$ and $W_- \circ R_-^{-1}$ is slowly varying at $0_-$, then* (I) *is equivalent to*

(II) $\left( 1 + \dfrac{W_-\left(R_-^{-1}(-x)\right)}{W_+\left(R_+^{-1}(x)\right)} \right)^{-1} = O(1)$ *as* $x \downarrow 0$.

*Proof.* In either of the three cases in (a), the spectrum of the operator $B_{+,0}$ is discrete and the eigenvalues accumulate at $+\infty$. This follows from the fact that in case (i) in (a) the operator $B_{+,0}$ is either regular or in the limit-circle case at $b_+$. In the remaining two cases in (a) this follows from Theorem 4.22. Similarly, in either of the three cases in (b), the spectrum of the operator $B_{-,0}$ is discrete and the eigenvalues accumulate at $\infty$. Since $A$ is a rank-one perturbation of the operator $B_{+,0} \oplus (-B_{-,0})$, by Weyl's theorem the spectrum of the operator $A$ is also discrete (see [66, Theorem XIII.14]). By Lemma 3.6 the eigenvalues of $A$ accumulate on both sides of $\infty$. Since the operator $A$ is nonnegative in the Krein space $\mathcal{K}$ all nonzero eigenvalues of $A$ are semi-simple and the length of the Jordan chain at $0$ is at most $2$. Moreover, by Lemma 4.3 all nonzero eigenvalues of $A$ are simple.

Let $\Delta$ be an arbitrary finite open interval such that $0 \in \Delta$ and let $E$ be the spectral function of $A$ in the sense of [59]. By the properties of this spectral function [59], $\infty \in c_r(A)$ if and only if there exists a Riesz basis of $(I - E(\Delta))\mathcal{K}$ which consists of eigenfunctions of the restriction of $A$ on $(I - E(\Delta))\mathcal{K}$. Since $E(\Delta)\mathcal{K}$ is a finite-dimensional space, the eigenfunctions and the generalized eigenfunctions of the restriction of $A$ on $E(\Delta)\mathcal{K}$ form a Riesz basis of $E(\Delta)\mathcal{K}$. Therefore (I) is equivalent to $\infty \in c_r(A)$. By Theorem 4.13, if either $W_+ \circ R_+^{-1}$ is positively increasing at $0_+$ or $W_- \circ R_-^{-1}$ is positively increasing at $0_-$, then $\infty \in c_r(A)$ and hence the claim in (A) holds.

If $W_+ \circ R_+^{-1}$ is slowly varying at $0_+$ and $W_- \circ R_-^{-1}$ is slowly varying at $0_-$, then by Theorem 4.17 condition (II) is equivalent to $\infty \in c_r(A)$. Since we already proved that $\infty \in c_r(A)$ is equivalent to (I), the equivalence in (B) is proved. $\square$

**Remark 4.27.** For the differential expression $\mathfrak{a}$ introduced in Example 4.21 we have

$$W_+(x)\big(R_+(1) - R_+(x)\big) = \frac{1-x}{(-\ln x)^{\alpha_+}} \sim (1-x)^{1-\alpha_+} \quad \text{as} \quad x \uparrow 1$$

and

$$W_-(x)(R_-(-1) - R_-(x)) = \frac{1+x}{\big(-\ln(-x)\big)^{\alpha_-}} \sim (1+x)^{1-\alpha_-} \quad \text{as} \quad x \downarrow -1.$$

Therefore $\mathfrak{a}$ satisfies conditions (a)(iii) and (b)(iii) in Theorem 4.26 if and only if $\alpha_- \in (0,1)$ and $\alpha_+ \in (0,1)$. By (B) in Theorem 4.26 the operator $A$ in Example 4.21 with $\alpha_-, \alpha_+ \in (0,1)$ has the Riesz basis property if and only if $\alpha_- \neq \alpha_+$.

4.5. **Regularity at $0$.** Since the operator $A$ associated with the differential expression $\mathfrak{a}$ is nonnegative it may have another critical point at $0$. In this subsection we consider the problem of regularity of the critical point $0$ of the operator $A$. Let $W_\pm$ and $R_\pm$ be defined by (4.16).

**Theorem 4.28.** *Let $W_\pm$ and $R_\pm$ be defined by* (4.16), *and let $A$ be the differential operator associated with the expression $\mathfrak{a}$ with the domain defined by* (4.13). *Assume that one of the following cases is in force:*

(i) $w_-, r_- \notin L^1(I_-)$, $w_+, r_+ \notin L^1(I_+)$ and either $W_- \circ R_-^{-1}$ is positively increasing at $-\infty$ or $W_+ \circ R_+^{-1}$ is positively increasing at $+\infty$;

(ii) $w_- \notin L^1(I_-)$ and $w_+ \notin L^1(I_+)$ and either $r_+ \in L^1(I_+)$, or $r_- \in L^1(I_-)$;

(iii) either $w_+ \in L^1(I_+)$, $w_- \notin L^1(I_-)$ or $w_- \in L^1(I_-)$, $w_+ \notin L^1(I_+)$.

(iv) $w_- \in L^1(I_-)$ and $w_+ \in L^1(I_+)$, and $W_+(b_+) + W_-(b_-) \neq 0$.

Then

$$0 \notin c_s(A) \quad and \quad \ker A = \ker A^2. \tag{4.49}$$

Moreover, the following statements hold.

(a) If $w_- \in L^1(I_-)$ and $w_+ \in L^1(I_+)$, then (4.49) holds if and only if $W_+(b_+) + W_-(b_-) \neq 0$.

(b) If $w_- \in L^1(I_-)$, $w_+ \in L^1(I_+)$ and (4.37), (4.42) hold, then $0 \notin \sigma_{\mathrm{ess}}(A)$ and the following three statements are equivalent

$$W_+(b_+) + W_-(b_-) \neq 0 \quad \Leftrightarrow \quad \ker A = \ker A^2 \quad \Leftrightarrow \quad 0 \notin c(A). \tag{4.50}$$

*Proof.* **1.** *Proof of* (4.49) *under assumption* (i). Due to Lemma 4.10 the assumption that $W_+ \circ R_+^{-1}$ is positively increasing at $+\infty$ is equivalent to the condition

$$\operatorname{Im} m_+(iy) = O\big(\operatorname{Re} m_+(iy)\big) \quad \text{as} \quad y \downarrow 0.$$

By Theorem 4.4 (v) this implies $0 \notin c_s(A)$ and $\ker A = \ker A^2$.

**2.** *Proof of* (4.49) *under assumption* (ii). If $r_+ \in L^1(\mathbb{R}_+)$ and $w_+ \notin L^1(\mathbb{R}_-)$, then by Lemma 4.11 (4.26) holds and, hence, by Theorem 4.4 (v) we have $0 \notin c_s(A)$ and $\ker A = \ker A^2$.

**3.** *Proof of* (4.49) *under assumption* (iii). If $w_+ \in L^1(I_+)$ and $w_- \notin L^1(I_-)$, then by Lemma 4.11

$$m_+(iy) = i\frac{a_+}{y} + \widetilde{m}_+(iy), \quad m_-(iy) = o(1/y) \quad \text{as} \quad y \downarrow 0$$

for $a_+ = \frac{1}{W_+(b_+)} > 0$, $\widetilde{m}_+(iy) = o(1/y)$. Then

$$m_+(iy) + m_-(-iy) \sim -\frac{a_+}{iy} \quad \text{as} \quad y \downarrow 0.$$

and

$$\operatorname{Im} m_+(iy) \sim \frac{a_+}{y}, \quad \operatorname{Im} m_-(iy) \to 0 \quad \text{as} \quad y \downarrow 0.$$

Hence

$$\operatorname{Im} m_\pm(iy) = O(m_+(iy) + m_-(-iy)) \quad \text{as} \quad y \downarrow 0$$

and by Theorem 4.4 (ii) $0 \notin c_s(A)$ and $\ker A = \ker A^2$.

**4.** *Proof of* (4.49) *under assumption* (iv). If $w_+ \in L^1(I_+)$ and $w_- \in L^1(I_-)$ then by Lemma 4.11

$$m_+(iy) \sim i\frac{a_+}{y}, \quad m_-(iy) \sim i\frac{a_-}{y} \quad \text{as} \quad y \downarrow 0$$

for $a_\pm = \pm 1/W_\pm(b_\pm)$. Since $W_+(b_+) + W_-(b_-) \neq 0$ then $a_+ \neq a_-$,

$$m_+(iy) + m_-(-iy) \sim i\frac{a_+ - a_-}{y} \quad \text{as} \quad y \downarrow 0$$

and

$$\operatorname{Im} m_\pm(iy) \sim \frac{a_\pm}{y} \quad \text{as} \quad y \downarrow 0. \tag{4.51}$$

Hence

$$\operatorname{Im} m_\pm(iy) = O\big(m_+(iy) + m_-(-iy)\big) \quad \text{as} \quad y \downarrow 0. \tag{4.52}$$

By Theorem 4.4 (ii) (4.52) is equivalent to $0 \notin c_s(A)$ and $\ker A = \ker A^2$.

**5.** *Proof of* (a). Assume now that $W_+(b_+) + W_-(b_-) = 0$. Then by Lemma 4.11 $a_+ = a_-$ and hence

$$m_+(iy) + m_-(-iy) = o(1/y) \quad \text{as} \quad y \downarrow 0.$$

In view of (4.51) the relation (4.52) is not fulfilled and by Theorem 4.4 the relations (4.49) fail to hold, i.e either $0 \in c_s(A)$ or $\ker A \subsetneq \ker A^2$.

**6.** *Proof of* (b). If $w_\pm \in L^1(I_\pm)$ and (4.37), (4.42) hold, then by Theorem 4.22 $0 \notin \sigma_{\mathrm{ess}}(B_{\pm,0})$. Since $A$ is a rank-one perturbation of the operator $B_{+,0} \oplus (-B_{-,0})$ we have $0 \notin \sigma_{\mathrm{ess}}(A)$.

Since $w \in L^1(I)$, all constant functions on $I$ belong to dom $A$ defined in (4.12). Consequently, all constant functions on $I$ belong to $\ker A$. As by Lemma 4.3 $\ker A$ is at most one-dimenional, we deduce that $\ker A$ consists of all the constant functions on $I$. Denote by $\mathbf{1}$ the constant function on $I$ equal to 1. Notice that

$$[\mathbf{1}, \mathbf{1}]_w = W_+(b_+) + W_-(b_-). \tag{4.53}$$

If $W_+(b_+) + W_-(b_-) \neq 0$ then the subspace $\ker A$ is nondegenerate. Moreover, we have $\ker A = \ker A^2$, since the existence of an associated vector $f \in \operatorname{dom} A$ such that $Af = \mathbf{1}$ implies $[\mathbf{1}, \mathbf{1}]_w = [Af, \mathbf{1}]_w = [f, A\mathbf{1}]_w = 0$. This proves the implication

$$W_+(b_+) + W_-(b_-) \neq 0 \quad \Rightarrow \quad \ker A = \ker A^2.$$

Now assume $\ker A = \ker A^2$. Then 0 is a simple eigenvalue of $A$ and since $0 \notin \sigma_{\mathrm{ess}}(A)$ it is an isolated eigenvalue. By [59] $\ker A$ is nondegenerate and thus $0 \notin c(A)$.

And finally, if $0 \notin c(A)$, then by [59] $\ker A^2 = \ker A$ and $\ker A$ is definite. Hence $[\mathbf{1}, \mathbf{1}]_w \neq 0$ and by (4.53) we have $W_+(b_+) + W_-(b_-) \neq 0$. This proves the implication $0 \notin c(A) \Rightarrow W_+(b_+) + W_-(b_-) \neq 0$ and hence the equivalence (4.50). $\square$

In Theorem 4.28 it is not claimed that $0 \in c_r(A)$, since it may happen that 0 is not a critical point of $A$ at all. In the next corollary we specify some cases when 0 is indeed a regular critical point of $A$.

**Corollary 4.29.** *Assume that $w_+$ and $r_+$ satisfy one of the following assumptions:*

(a) $w_+ \in L^1(I_+)$, $r_+ \notin L^1(I_+)$ *and* $\sup_{x \in I_+} R_+(x)\big(W_+(b_+) - W_+(x)\big) = +\infty$,

(b) $w_+ \notin L^1(I_+)$, $r_+ \in L^1(I_+)$ *and* $\sup_{x \in I_+} W_+(x)\big(R_+(b_+) - R_+(x)\big) = +\infty$.

*Assume that $w_-$ and $r_-$ satisfy one of the following assumptions:*

(c) $w_- \in L^1(I_-)$, $r_- \notin L^1(I_-)$ *and* $\sup_{x \in I_-} R_-(x)\big(W_-(b_-) - W_-(x)\big) = +\infty$,

(d) $w_- \notin L^1(I_-)$, $r_- \in L^1(I_-)$ *and* $\sup_{x \in I_-} W_-(x)\big(R_-(b_-) - R_-(x)\big) = +\infty$.

*In cases* (a) *and* (c) *assume* $W_+(b_+) + W_-(b_-) \neq 0$. *Then* $0 \in c_r(A)$ *and the spectrum of the operator $A$ accumulates on both sides of* 0.

*Proof.* In either of the cases (a) and (b) ((c) and (d), respectively), 0 is an accumulation point for the spectrum of the operator $B_{+,0}$ ($B_{-,0}$, respectively) from the right. Therefore, 0 is an accumulation point for the spectrum of the decoupled operator $A_0 = A_{+,0} \oplus (A_{-,0})$ from both sides. Since the resolvent $(A - z)^{-1}$ of $A$

is a one-dimensional perturbation of the resolvent $(A_0 - z)^{-1}$, see (3.7), it follows from [42, Theorem 1] that $0 \in c(A)$.

The statement $0 \in c_r(A)$ follows from Theorem 4.28. $\qquad\square$

**Remark 4.30.** The list of assumptions of Theorem 4.28 covers all possible cases except the following:

(v) $w_-, r_- \notin L^1(I_-)$, $w_+, r_+ \notin L^1(I_+)$ and both $W_- \circ R_-^{-1}$ is not positively increasing at $-\infty$ and $W_+ \circ R_+^{-1}$ is not positively increasing at $+\infty$.

In this case we cannot apply our abstract results from Theorem 3.10 because the asymptotic behaviour of the Weyl functions at finite points is insufficiently studied.

**Remark 4.31.** If $w_+ \in L^1(I_+)$, $w_- \in L^1(I_-)$, $R_+ \in L^1_{w_+}(I_+)$ and $R_- \in L^1_{w_-}(I_-)$ then by Proposition 4.25 the spectrum of $A$ is discrete and in the case $W_+(b_+) + W_-(b_-) = 0$ the root subspace $\ker A^2$ can be found explicitly. As was mentioned above $\ker A = \mathrm{span}\{\mathbf{1}\}$. Let us find a generalized eigenvector $f \in \mathrm{dom}(A)$ such that $Af = \mathbf{1}$, i.e. $f = f_+ \oplus f_-$, where $f_\pm \in \mathrm{dom}(B_\pm^{\langle * \rangle})$ are solutions of the equations

$$\mathfrak{b}_+ f_+ = 1, \quad -\mathfrak{b}_- f_- = 1, \tag{4.54}$$

such that

$$f_+(0) = f_-(0), \quad f_+^{[1]}(0) = f_-^{[1]}(0). \tag{4.55}$$

holds. Straightforward calculations show that the functions

$$f_\pm(x) = \pm \int_0^x R_\pm(\xi) w_\pm(\xi) d\xi \pm \int_x^{b_\pm} R_\pm(x) w_\pm(\xi) d\xi \tag{4.56}$$

satisfy (4.54) and the first boundary condition in (4.55). The second boundary condition in (4.55) holds since $W_+(b_+) + W_-(b_-) = 0$.

It follows from (4.48) that the second term in (4.56)

$$\int_x^{b_\pm} R_\pm(x) w_\pm(\xi) d\xi = R_\pm(x)(W_\pm(b_\pm) - W_\pm(x))$$

is bounded. The first term in the right hand part of (4.56) is also bounded since $R_\pm \in L^1_{w_\pm}(I_\pm)$ and hence $f_\pm \in L^2_{w_\pm}(I_\pm)$.

Therefore, $f_\pm \in \mathrm{dom}(B_{\pm,\max})$ and hence $f_\pm \in \mathrm{dom}(B_\pm^{\langle * \rangle})$ in the limit point case. In the limit circle case we also get $f_\pm \in \mathrm{dom}(B_\pm^{\langle * \rangle})$, since $f_\pm^{[1]}(b_\pm) = 0$. Therefore, $f = f_+ \oplus f_- \in \mathrm{dom}\, A$ and the equation $Af = \mathbf{1}$ has a solution $f \in \mathrm{dom}(A)$. Thus $\ker A \neq \ker A^2$.

**Remark 4.32.** Let $b_+, \alpha, \beta, b_-$ and the function $w$ be defined as in Corollary 4.19 and arbitrary $r \in L^1_{\mathrm{loc}}(I)$. Assume that $w_+ \in L^1(I_+)$. Then $w_- \in L^1(I_-)$. Let $W_+$ be the function defined in (4.16). Then, for all $x \in [b_-, 0]$ we have $W_-(x) = -(\alpha/\beta)W_+(-\beta x)$. Consequently,

$$W_+(b_+) + W_-(b_-) = (1 - \alpha/\beta)\, W_+(b_+).$$

By Theorem 4.28 (a) we have that (4.49) holds if and only if $\alpha \neq \beta$. In particular, if $\alpha = \beta = 1$ the weight function $w(t)$ is odd and condition (4.49) does not hold. This has been proved in [58, Theorem 4.7] under additional conditions that $r$ is even and $r \notin L^1(I)$.

**Example 4.33.** We consider the differential expression studied in Example 4.21 on an interval $I = (b_-, b_+)$ with $-1 \leq b_- < 0 < b_+ \leq 1$.

First assume that $b_- = -1$ and $b_+ = 1$, as in Example 4.21. Then $R_\pm$ and $W_\pm$ are given by the formulas (4.34) and (4.35). Due to Theorem 4.28 (ii) $0 \notin c_s(A)$ and $\ker A = \ker A^2$.

Moreover, $\ker(A) = \{0\}$ since a function $f \in \ker(A)$ should have a form $f = f_+ \oplus f_- \in \operatorname{dom} A$, where $f_+ \in \operatorname{dom}(B_+^{\langle * \rangle})$, $f_- \in \operatorname{dom}(B_-^{\langle * \rangle})$ and the coupling conditions (4.55) hold. The conditions $f_+ \in \operatorname{dom}(B_+^{\langle * \rangle})$, $f_- \in \operatorname{dom}(B_-^{\langle * \rangle})$ yield that $f_+$ and $f_-$ are proportional to $1 - x$ and $1 + x$, respectively. But then the coupling conditions (4.55) yield $f_+ = f_- = 0$.

Further,

$$\lim_{x \uparrow 1} \big( R_+(1) - R_+(x) \big) W_+(x) = \lim_{x \uparrow 1} \frac{1 - x}{(-\ln x)^{\alpha_+}} = \begin{cases} 0 & \text{if} \quad 0 < \alpha_+ < 1, \\ 1 & \text{if} \quad \alpha_+ = 1, \\ +\infty & \text{if} \quad \alpha_+ > 1, \end{cases}$$

and

$$\lim_{x \downarrow -1} \big( R_-(-1) - R_-(x) \big) W_-(x) = \lim_{x \downarrow -1} \frac{1 + x}{\big( -\ln(-x) \big)^{\alpha_-}} = \begin{cases} 0 & \text{if} \quad 0 < \alpha_- < 1, \\ 1 & \text{if} \quad \alpha_- = 1, \\ +\infty & \text{if} \quad \alpha_- > 1. \end{cases}$$

Hence Corollary 4.23 yields

$$0 \in \sigma_{\text{ess}}(B_{-,0}) \cap \sigma_{\text{ess}}(B_{+,0}) \quad \Leftrightarrow \quad \alpha_+ > 1 \quad \text{and} \quad \alpha_- > 1.$$

Since $0$ is not an eigenvalue of $A$, it follows from the preceding equivalence that $0 \in c(A)$ if and only if $\alpha_+ > 1$ and $\alpha_- > 1$. Theorem 4.28 (ii) yields that $0 \in c_r(A)$, whenever $\alpha_+ > 1$ and $\alpha_- > 1$. Conversely, if $\alpha_+ \in (0, 1]$ or $\alpha_- \in (0, 1]$, then $0 \notin c(A)$. Consequently, $0 \in c_r(A)$ if and only if $\alpha_+ > 1$ and $\alpha_- > 1$.

Next assume that $b_+ = 1$ and $b_- \in (-1, 0)$. Due to Theorem 4.28 (iii) $0 \notin c_s(A)$. In this case $\ker(A) = \{0\}$, since a function $f \in \ker(A)$ should have a form $f = f_+ \oplus f_- \in \operatorname{dom} A$, where $f_+ \in \operatorname{dom}(B_+^{\langle * \rangle})$, $f_- \in \operatorname{dom}(B_-^{\langle * \rangle})$ and satisfy the conditions

$$f_+(0) = f_-(0), \quad f_+'(0) = f_-'(0), \quad f_-'(b_-) = 0. \tag{4.57}$$

The conditions $f_+ \in \operatorname{dom}(B_+^{\langle * \rangle})$ and $f_-'(b_-) = 0$ yield that $f_+$ is proportional to $1 - x$ and $f_-$ is constant. Then (4.57) implies $f_+ = 0$ and hence, also $f_- = 0$.

Since the spectrum of $B_{-,0}$ is discrete $0$ is not an accumulation point of the negative spectrum of $A$ and consequently $0 \notin c(A)$. The same conclusion holds if $b_- = -1$ and $b_+ \in (0, 1)$.

Finally we assume that $b_- \in (-1, 0)$ and $b_+ \in (0, 1)$. In this case the differential expression $\mathfrak{a}$ is regular, so the spectrum of $A$ is discrete. Therefore the root space at $0$ is nondegenerate. Consequently, $0 \notin c_s(A)$. Since

$$W_-(b_-) + W_+(b_+) = \frac{1}{(-\ln b_+)^{\alpha_+}} - \frac{1}{(-\ln |b_-|)^{\alpha_-}},$$

statement (b) from Theorem 4.28 takes the form:

$$\frac{\alpha_+}{\alpha_-} \neq \frac{\ln \big| \ln |b_-| \big|}{\ln |\ln b_+|} \quad \Leftrightarrow \quad \ker A = \ker A^2 \quad \Leftrightarrow \quad 0 \notin c(A).$$

It is interesting to write the preceding equivalences in the following form:

$$\frac{\alpha_+}{\alpha_-} = \frac{\ln\big|\ln|b_-|\big|}{\ln|\ln b_+|} \quad \Leftrightarrow \quad \ker A \subsetneq \ker A^2 \quad \Leftrightarrow \quad 0 \in c_r(A).$$

4.6. **Similarity.** The coupling operator $A$ in the Krein space $\mathcal{K}$ is nonnegative and $\rho(A) \neq \emptyset$. Hence, it has at most two critical points 0 and $\infty$. Thus, $A$ is similar to a self-adjoint operator in a Hilbert space if and only if its critical points are regular and $\ker A = \ker A^2$, see Theorem 2.2. Combining Theorems 4.4, 4.28, 4.13, and 4.17 we obtain the following list of sufficient conditions for similarity of $A$ to a self-adjoint operator in a Hilbert space, which equals the Property (Si) from the introduction.

**Theorem 4.34.** *Let $A$ be the differential operator associated with the expression $\mathfrak{a}$ with the domain defined by (4.13) and let $W_\pm$ and $R_\pm$ be defined by (4.16). Let at least one of the conditions (i)-(iv) in Theorem 4.28 be in force. Then the following statements hold.*

(a) *If either $W_+ \circ R_+^{-1}$ is positively increasing at $0_+$ or $W_- \circ R_-^{-1}$ is positively increasing at $0_-$, then the operator $A$ is similar to a self-adjoint operator in a Hilbert space.*

(b) *Let $W_+ \circ R_+^{-1}$ be slowly varying at $0_+$ and let $W_- \circ R_-^{-1}$ be slowly varying at $0_-$. Then the operator $A$ is similar to a self-adjoint operator in a Hilbert space if and only if*

$$\left(1 + \frac{W_-\big(R_-^{-1}(-x)\big)}{W_+\big(R_+^{-1}(x)\big)}\right)^{-1} = O(1) \quad as \quad x \downarrow 0.$$

**Example 4.35.** Let us consider Example 4.21 on an interval $I = (b_-, b_+)$ with $-1 \leq b_- < 0 < b_+ \leq 1$. Combining the conclusions made in Example 4.21 and Example 4.33 we obtain the following equivalence:

The operator $A$ is similar to a self-adjoint operator in a Hilbert space if and only if

(1) either $\max\{b_+, |b_-|\} = 1$ and $\frac{\alpha_+}{\alpha_-} \neq 1$;

(2) or $\max\{b_+, |b_-|\} < 1$ and $\frac{\alpha_+}{\alpha_-} \notin \left\{1, \frac{\ln|\ln|b_-||}{\ln|\ln b_+|}\right\}$.

## Appendix A. Some results from Karamata's theory

In Appendix we present the definitions and the results from Karamata's theory of regularly varying functions that we use in the paper. Standard references for Karamata's theory are [10] and [71]. For completeness we include a few standard results from Karamata's theory and some of these results are reformulated to fit our needs. In addition, we present Theorem A.7, Corollary A.8, and the results of Subsection A.3 that seem to be new.

A.1. **Definitions and basic results.** First we give definitions of regularly varying functions.

**Definition A.1.** Let $a, \alpha \in \mathbb{R}$ with $a > 0$. A measurable function $f : (0, a] \to \mathbb{R}_+$ is called *regularly varying at 0 from the right with index $\alpha$* if the following condition is satisfied:

$$\text{for all} \quad \lambda \in \mathbb{R}_+ \qquad \text{we have} \qquad \lim_{x \downarrow 0} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha.$$

When $\alpha = 0$ the function $f$ is called *slowly varying at 0 from the right*.

A measurable function $g : [a, +\infty) \to \mathbb{R}_+$ is called *regularly varying at $+\infty$ with index $\alpha$* if the following condition is satisfied:

$$\text{for all} \quad \lambda \in \mathbb{R}_+ \qquad \text{we have} \qquad \lim_{x \to +\infty} \frac{g(\lambda x)}{g(x)} = \lambda^\alpha.$$

When $\alpha = 0$ the function $g$ is called *slowly varying at $+\infty$*.

A measurable function $g : [-a, 0) \to \mathbb{R}_-$ is called *regularly varying at $0$ from the left with index $\alpha$* if the function $f(x) = -g(-x)$ where $x \in (0, a]$ is regularly varying at $0$ from the right with index $\alpha$. When $\alpha = 0$ the function $g$ is called *slowly varying at $0$ from the left*.

We will often use "at $0_+$" as an abbreviation for the phrase "at $0$ from the right" and "at $0_-$" as an abbreviation for the phrase "at $0$ from the left."

The Karamata's theory of regular variation is commonly presented for functions regularly varying at $+\infty$. The results for functions regularly varying at $0_+$ follow from the following equivalence. Let $f$ and $g$ be measurable functions such that $g(x) = f(1/x)$ for all $x$ in the domain of $g$ for which $1/x$ is in the domain of $f$. Then $g$ is regularly varying at $+\infty$ with index $\alpha$ if and only if $f$ is regularly varying at $0_+$ with index $-\alpha$.

In this section some results will be presented at $0_+$ and some at $+\infty$. This choice is sometimes made based on our needs in this paper and sometimes on convenience.

Slow variation plays the central role in the theory of regular variation. That centrality is expressed in the following proposition that follows immediately from the definition.

**Proposition A.2.** *Let $a, \alpha \in \mathbb{R}$ with $a > 0$ and let $f, g : (0, a] \to \mathbb{R}_+$ be measurable functions such that $g(x) = x^\alpha f(x)$ for all $x \in (0, a]$. The function $g$ is regularly varying at $0_+$ with index $\alpha$ if and only if $f$ is slowly varying at $0_+$.*

The next proposition is a simple characterization of slowly varying functions which is most conveniently stated at $+\infty$. A simple proof is similar to the proof of Proposition A.16 at the end of this section.

**Proposition A.3.** *Let $a \in \mathbb{R}_+$ and let $f : [a, +\infty) \to \mathbb{R}_+$ be a measurable function. Set $b = \ln a$ and set $\phi$ to be the composition $\ln \circ f \circ \exp$ defined on $[b, +\infty)$. The function $f$ is a slowly varying function at $+\infty$ if and only if the following condition is satisfied:*

$$\text{for all} \quad c \in \mathbb{R} \qquad \text{we have} \qquad \lim_{u \to +\infty} \big(\phi(u) - \phi(u - c)\big) = 0.$$

The next theorem is Karamata's Representation Theorem, see [10, Theorem 1.3.1] or [53] for Karamata's original paper.

**Theorem A.4.** *Let $a \in \mathbb{R}$. A function $f : [a, +\infty) \to \mathbb{R}_+$ is slowly varying at $+\infty$ if and only if there exist $b \in [a, +\infty)$, a measurable function $m : [b, +\infty) \to \mathbb{R}_+$ and a continuous function $\varepsilon : [b, +\infty) \to \mathbb{R}$ such that*

$$\lim_{x \to +\infty} m(x) = M \in \mathbb{R}_+, \qquad \lim_{x \to +\infty} \varepsilon(x) = 0,$$

*and for all $x \geq b$ we have*

$$f(x) = m(x) \exp \left( \int_b^x \frac{\varepsilon(t)}{t} dt \right). \tag{A.1}$$

The following property of regularly varying functions follows from Proposition A.2 and Theorem A.4, see [71, 1° on page 18].

**Corollary A.5.** *If $g$ is a regularly varying function at $+\infty$ with a positive (negative, respectively) index, then*

$$\lim_{x \to +\infty} g(x) = +\infty \qquad (\lim_{x \to +\infty} g(x) = 0, \text{ respectively}).$$

*If $f$ is a regularly varying function at $0_+$ with a positive (negative, respectively) index, then*

$$\lim_{x \downarrow 0} f(x) = 0 \qquad (\lim_{x \downarrow 0} f(x) = +\infty, \text{ respectively}).$$

If the measurable function $m$ in (A.1) can be chosen to be constant, then $f$ is called a *normalized slowly varying* function at $+\infty$. The following characterization is deduced from Theorem A.4: A continuously differentiable function $f : [a, +\infty) \to \mathbb{R}_+$ is a normalized slowly varying function at $+\infty$ if and only if

$$\frac{xf'(x)}{f(x)} = o(1) \quad \text{as} \quad x \to +\infty, \tag{A.2}$$

see [71, page 7]. Consequently, the natural logarithm $\ln x$ is a normalized slowly varying function at $+\infty$. In fact the natural logarithm is the most popular slowly varying function at $+\infty$, while the natural logarithm composed with itself $\ln_2 x := \ln(\ln x), x \in [e, +\infty)$, is the close second. We utilize them both in the next paragraph.

Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a continuously differentiable function. A convenient way to construct normalized slowly varying functions at $+\infty$ is to compose a function of the form $\exp\big(\phi(u)\big)$ with $u = \ln x$ or $u = \ln_2 x$. The following characterizations of the normalized slowly varying (nsv) functions at $+\infty$ are easily deduced from the characterization (A.2):

$$\exp\big(\phi\big(\ln x\big)\big) \text{ is nsv at } +\infty \quad \Leftrightarrow \quad \phi'(u) = o(1) \text{ as } u \to +\infty, \tag{A.3}$$

$$\exp\big(\phi\big(\ln_2 x\big)\big) \text{ is nsv at } +\infty \quad \Leftrightarrow \quad \phi'(u) = o\big(\exp(u)\big) \text{ as } u \to +\infty. \tag{A.4}$$

To illustrate the power of these characterizations we use (A.4) to present an example of a normalized slowly varying function with infinite oscillations. A basic example of a function with infinite oscillations is

$$\phi(u) = u \cos u \quad \text{where} \quad u \in \mathbb{R}_+.$$

Since clearly $\phi'(u) = o\big(\exp(u)\big)$ as $u \to +\infty$, characterization (A.4) yields that the function

$$f(x) = \exp\big((\ln_2 x) \cos(\ln_2 x)\big) \quad \text{where} \quad x \in [e, +\infty),$$

is a normalized slowly varying function at $+\infty$. Since

$$f\Big(e^{e^{(2k-1)\pi}}\Big) = e^{-(2k-1)\pi} \quad \text{and} \quad f\Big(e^{e^{2k\pi}}\Big) = e^{2k\pi} \quad \text{where} \quad k \in \mathbb{N},$$

we can informally say that the function $f$ oscillates between 0 and $+\infty$, that is $f$ exhibits infinite oscillations. Notice that a more complicated example

$$f(x) = \exp\Big(\big(\sqrt[3]{\ln x}\big) \cos\big(\sqrt[3]{\ln x}\big)\Big) \quad \text{where} \quad x \in [1, +\infty),$$

appears often in literature, as for example in [10, page 16]. That the preceding function is a normalized slowly varying function at $+\infty$ is deduced from (A.3). The cube root in the last displayed formula cannot be replaced by the square root

since for such a function the right-hand side of (A.3) does not hold. This was overlooked in [78, (A.5) on page 570].

A.2. **Karamata's characterization and consequences.** The following theorem is our restatement of Karamata's characterization of regular variation as it appears in [38, Theorem 1.2.1], [56, Theorems IV.5.2 and IV.5.3], [10, Theorems 1.5.11 and 1.6.1] and [12]. In [10, 12, 38, 56] regular variation at $+\infty$ is considered. Here we characterize regular variation at $0_+$.

**Theorem A.6.** *Let $a \in \mathbb{R}_+$ and let $f : (0, a] \to \mathbb{R}_+$ be a locally integrable function on $(0, a]$. Let $\alpha, \gamma \in \mathbb{R}$ be such that $\gamma + \alpha \neq 0$ and consider the following two conditions:*

$$\int_0^a s^{\gamma-1} f(s) ds \quad \text{exists as an improper integral at } 0, \tag{A.5}$$

$$\lim_{v \downarrow 0} \frac{1}{v^\gamma f(v)} \int_0^v s^{\gamma-1} f(s) ds = \frac{1}{\gamma + \alpha}. \tag{A.6}$$

*The following statements are equivalent:*

(a) *$f$ is regularly varying at $0_+$ with index $\alpha$.*
(b) *For all $\gamma \in \mathbb{R}$ such that $\gamma + \alpha > 0$ conditions (A.5) and (A.6) hold.*
(c) *There exists $\gamma \in \mathbb{R}$ such that $\gamma + \alpha > 0$ and conditions (A.5) and (A.6) hold.*

The next theorem is a reformulation of the preceding one in terms of the differential of the function under consideration.

**Theorem A.7.** *Let $a, \alpha, \gamma \in \mathbb{R}$ be such that $a > 0$, $\gamma \neq 0$ and $\gamma + \alpha \neq 0$. Let $f : (0, a] \to \mathbb{R}_+$ be a measurable function which is of bounded variation on each closed interval contained in $(0, a]$. Consider the following three conditions:*

$$\int_0^a s^\gamma df(s) \quad \text{exists as an improper Riemann-Stieltjes integral at } 0, \tag{A.7}$$

$$\lim_{v \downarrow 0} v^\gamma f(v) = 0, \tag{A.8}$$

$$\lim_{v \downarrow 0} \frac{1}{v^\gamma f(v)} \int_0^v s^\gamma df(s) = \frac{\alpha}{\gamma + \alpha}. \tag{A.9}$$

*The following statements are equivalent:*

(i) *$f$ is regularly varying at $0_+$ with index $\alpha$.*
(ii) *For all $\gamma \in \mathbb{R} \backslash \{0\}$ such that $\gamma + \alpha > 0$ conditions (A.7), (A.8) and (A.9) hold.*
(iii) *There exists $\gamma \in \mathbb{R} \backslash \{0\}$ such that $\gamma + \alpha > 0$ and conditions (A.7), (A.8) and (A.9) hold.*

*Proof.* Let $u, v \in (0, a]$ such that $u < v$. First notice that since $f$ is of bounded variation on $[u, v]$, see [77, Theorems 2.21 and 2.24], the integration by parts yields

$$\int_u^v s^\gamma df(s) = v^\gamma f(v) - u^\gamma f(u) - \gamma \int_u^v s^{\gamma-1} f(s) ds. \tag{A.10}$$

Assume (i). Let $\gamma \in \mathbb{R} \backslash \{0\}$ be such that $\gamma + \alpha > 0$. Since by Definition A.1 the function $x \mapsto x^\gamma f(x)$ is regularly varying at $0_+$ with index $\gamma + \alpha$, Corollary A.5 yields (A.8).

Theorem A.6 implies that (A.5) and (A.6) hold. Letting $u \downarrow 0$ in (A.10) and using (A.5) yields (A.7) and

$$\frac{1}{v^\gamma f(v)} \int_0^v s^\gamma df(s) = 1 - \frac{\gamma}{v^\gamma f(v)} \int_0^v s^{\gamma-1} f(s) ds. \tag{A.11}$$

Now letting $v \downarrow 0$ and using (A.6) we deduce (A.9), proving (ii).

The fact that (ii) implies (iii) is trivial. Now assume (iii). Letting $u \downarrow 0$ in (A.10) and using (A.7) yields (A.5), and we again deduce (A.11). Together (A.11) and (A.9) imply (A.6) in Theorem A.6. Thus, (c) in Theorem A.6 holds and (i) follows from Theorem A.6. □

Let $\gamma > 0$. With the substitution $t = v^\gamma$, conditions (A.7), (A.8) and (A.9) can be rewritten as (see [63, Theorem 12.11] for the change of variables formula in Riemann-Stieltjes integral)

$$\int_0^{a^\gamma} t \, df(t^{1/\gamma}) \quad \text{exists as an improper Riemann-Stieltjes integral at 0,}$$

$$\lim_{t \downarrow 0} t f(t^{1/\gamma}) = 0,$$

$$\lim_{t \downarrow 0} \frac{1}{t f(t^{1/\gamma})} \int_0^t s \, df(s^{1/\gamma}) = \frac{\alpha/\gamma}{1 + \alpha/\gamma}.$$

This observation and Theorem A.7 (with $\gamma$ being 1 and $\alpha$ being $\alpha/\gamma$) yield the following equivalence: The function $t \mapsto f(t^{1/\gamma})$ with $t \in (0, a^\gamma]$ is regularly varying at $0_+$ with index $\alpha/\gamma > -1$ if and only if conditions (A.7), (A.8), (A.9) hold. Here it is convenient to read the last fraction in (A.9) as $(\alpha/\gamma)/(1 + (\alpha/\gamma))$.

The next corollary generalizes the preceding equivalence to any increasing bijection on $[0, a]$.

**Corollary A.8.** *Let $\alpha, a, b \in \mathbb{R}$ be such that $a, b > 0$ and $\alpha > -1$. Let $f : (0, b] \to \mathbb{R}_+$ be a function of bounded variation on every closed subinterval of $(0, b]$ and let $g : [0, b] \to [0, a]$ be an increasing bijection. The function $f \circ g^{-1} : (0, a] \to \mathbb{R}_+$ is regularly varying at $0_+$ with index $\alpha > -1$ if and only if the following three conditions are satisfied:*

$$\int_0^b g(s) df(s) \quad \text{exists as an improper Riemann-Stieltjes integral at 0,} \tag{A.12}$$

$$\lim_{v \downarrow 0} f(v) g(v) = 0, \tag{A.13}$$

$$\lim_{v \downarrow 0} \frac{1}{f(v) g(v)} \int_0^v g(s) df(s) = \frac{\alpha}{1 + \alpha}. \tag{A.14}$$

*Proof.* Let $u, v \in (0, b]$ such that $u < v$. As in the preceding theorem we notice that since $f$ is of bounded variation on $[u, v]$ the integration by parts ([77, Theorem 2.21]) yields

$$\int_u^v g(s) df(s) = f(v) g(v) - f(u) g(u) - \int_u^v f(s) dg(s). \tag{A.15}$$

In this proof we will also use that, since $g$ is a continuous increasing bijection, we have that $u \downarrow 0$ if and only if $g(u) \downarrow 0$.

Assume (A.12), (A.13) and (A.14). Letting $u \downarrow 0$ and using (A.12) and (A.13) in (A.15) yields

$$\int_0^v g(s)\,df(s) = f(v)g(v) - \int_0^v f(s)\,dg(s) \tag{A.16}$$

for all $v \in (0, b]$. Therefore, for all $v \in (0, b]$ we have

$$\frac{1}{f(v)g(v)}\int_0^v g(s)\,df(s) = 1 - \frac{1}{f(v)g(v)}\int_0^v f(s)dg(s)$$
$$= 1 - \frac{1}{f(v)g(v)}\int_0^{g(v)} f\big(g^{-1}(t)\big)dt, \tag{A.17}$$

where, for the second equality, we used the change of variables formula in Riemann-Stieltjes integral, [63, Theorem 12.11]. Now (A.14) implies

$$\frac{1}{1+\alpha} = \lim_{v \downarrow 0} \frac{1}{f(v)g(v)}\int_0^{g(v)} f\big(g^{-1}(t)\big)dt = \lim_{u \downarrow 0} \frac{1}{uf\big(g^{-1}(u)\big)}\int_0^u f\big(g^{-1}(t)\big)dt.$$

Since we assume $1 + \alpha > 0$, Theorem A.6 yields that $f \circ g^{-1}$ is regularly varying at $0_+$ with index $\alpha$.

To prove the converse assume that $f \circ g^{-1}$ is regularly varying at $0_+$ with index $\alpha > -1$. Then the function $x \mapsto xf\big(g^{-1}(x)\big)$ is regularly varying at $0_+$ with index $\alpha + 1 > 0$ and (A.13) follows from Corollary A.5 after a change of variables in the limit. By the change of variables formula for all $u \in (0, a]$ we have

$$\int_u^a s\,df\big(g^{-1}(s)\big) = \int_{g^{-1}(u)}^b g(t)\,df(t).$$

Consequently, (A.12) follows from (A.7) in Theorem A.7 applied to $f \circ g^{-1}$ with $\gamma = 1$. Therefore, (A.16) and consequently (A.17) both hold. Now (A.14) follows from (A.6) in Theorem A.6 with $\gamma = 1$. $\qquad\square$

A.3. **Asymptotic equivalence of functions on a sequence.** In the next definition we extend the notation $\sim$ of asymptotic equivalence of functions to hold only on a sequence.

**Definition A.9.** Let $a \in \mathbb{R}_+$. For functions $f, g : [a, +\infty) \to \mathbb{R}_+$ we write

$$f \overset{s}{\sim} g \quad \text{at} \quad +\infty$$

if and only if there exists an increasing sequence $(x_n)$ in $[a, +\infty)$ such that

$$\lim_{n \to +\infty} x_n = +\infty \quad \text{and} \quad \lim_{n \to +\infty} \frac{f(x_n)}{g(x_n)} = 1. \tag{A.18}$$

For functions $f, g : (0, a] \to \mathbb{R}_+$ we write

$$f \overset{s}{\sim} g \text{ at } 0_+$$

if and only if there exists a decreasing sequence $(x_n)$ in $(0, a]$ such that

$$\lim_{n \to +\infty} x_n = 0 \quad \text{and} \quad \lim_{n \to +\infty} \frac{f(x_n)}{g(x_n)} = 1.$$

**Proposition A.10.** *Let $f$ and $g$ be regularly varying functions at $+\infty$ with indices $\alpha$ and $\beta$, respectively. If $f \overset{s}{\sim} g$ at $+\infty$, then $\alpha = \beta$.*

*Proof.* We will prove the contrapositive. Assume that $\alpha < \beta$. Since the function $f(x)/g(x)$ is regularly varying with index $\alpha - \beta < 0$ it follows from Corollary A.5 that $\lim_{x \to +\infty} f(x)/g(x) = 0$. Thus, $f \overset{s}{\sim} g$ at $+\infty$ is not true. If $\alpha > \beta$ the preceding limit is $+\infty$, so $f \overset{s}{\sim} g$ at $+\infty$ is not true in this case either. $\qquad \square$

The converse of the preceding proposition is not true. For example, let $f$ be a slowly varying function at $+\infty$ and $g = 2f$. Then $\alpha = \beta = 0$, but $f \overset{s}{\sim} g$ at $+\infty$ is clearly not true.

The following theorem extends [68, Proposition 0.8(vi)] to the concept introduced in the previous definition.

**Theorem A.11.** *Let $f$ and $g$ be strictly monotonic positive functions defined in a neighbourhood of $+\infty$ and let $f$ be regularly varying at $+\infty$ with a nonzero index.*

(a) *If $f$ and $g$ are increasing and unbounded, then the inverses $f^{-1}$ and $g^{-1}$ are also increasing and unbounded and the following equivalence holds*

$$f \overset{s}{\sim} g \ at \ +\infty \quad \Leftrightarrow \quad f^{-1} \overset{s}{\sim} g^{-1} \ at \ +\infty.$$

(b) *If $f$ and $g$ are both decreasing with $0$ as the limit, then the inverses $f^{-1}$ and $g^{-1}$ are decreasing functions defined in a neighbourhood of $0_+$ and the following equivalence holds*

$$f \overset{s}{\sim} g \ at \ +\infty \quad \Leftrightarrow \quad f^{-1} \overset{s}{\sim} g^{-1} \ at \ 0_+.$$

*Proof.* (a) Assume that $f$ and $g$ are increasing and unbounded and that $f$ is regularly varying at $+\infty$ with positive index $\alpha$. Assume also that $f \overset{s}{\sim} g$ at $+\infty$ and let $(x_n)$ be an increasing sequence in the common domain of $f$ and $g$ such that (A.18) holds. For all $n \in \mathbb{N}$ set $y_n = f(x_n)$ and $z_n = g(x_n)$. Then, for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ we have

$$n > N_\epsilon \quad \Rightarrow \quad 1 - \epsilon < \frac{y_n}{z_n} < 1 + \epsilon. \tag{A.19}$$

Clearly the inverses $f^{-1}$ and $g^{-1}$ are increasing functions defined in a neighbourhood of $+\infty$. It follows from [10, Theorem 1.5.12] that $f^{-1}$ is regularly varying function at $+\infty$ with the index $1/\alpha \in \mathbb{R} \setminus \{0\}$. By the definition of regular variation for all $\lambda \in \mathbb{R}_+$ we have

$$\lim_{y \to +\infty} \frac{f^{-1}(\lambda y)}{f^{-1}(y)} = \lambda^{1/\alpha}. \tag{A.20}$$

Denote by $\Delta$ the closed interval with the endpoints $(1/2)^{1/\alpha}$ and $(3/2)^{1/\alpha}$. By the Karamata Uniform Convergence Theorem for regularly varying functions, see [10, Theorem 1.5.2], the convergence in (A.20) is uniform in $\lambda$ while $\lambda \in \left[\frac{1}{2}, \frac{3}{2}\right]$. Since the function $u \mapsto u^\alpha$ is uniformly continuous on a closed interval which strictly contains $\Delta$, we deduce that

$$\lim_{y \to +\infty} \left( \frac{f^{-1}(\lambda y)}{f^{-1}(y)} \right)^\alpha = \lambda \qquad \text{uniformly in } \lambda \text{ on } \left[\tfrac{1}{2}, \tfrac{3}{2}\right].$$

By the definition of the uniform limit for every $\epsilon > 0$ there exists $Y_\epsilon > 0$ such that for all $y > Y_\epsilon$ and for all $\lambda \in \left[\frac{1}{2}, \frac{3}{2}\right]$ we have

$$\lambda - \epsilon < \left( \frac{f^{-1}(\lambda y)}{f^{-1}(y)} \right)^\alpha < \lambda + \epsilon. \tag{A.21}$$

Since $z_n = g(x_n) \to +\infty$ as $n \to +\infty$, for every $\epsilon > 0$ there exists $M_\epsilon \in \mathbb{N}$ such that for all $n > M_\epsilon$ we have $z_n > Y_\epsilon$.

Let $\epsilon \in \left(0, \frac{1}{2}\right]$ and let $n \in \mathbb{N}$ be such that $n > \max\{N_\epsilon, M_\epsilon\}$. Then, since $n > N_\epsilon$, by (A.19) we have

$$\frac{y_n}{z_n} \in (1 - \epsilon, 1 + \epsilon) \subset \left[\frac{1}{2}, \frac{3}{2}\right],$$

and since $n > M_\epsilon$ we have $z_n > Y_\epsilon$. Now, substituting $y = z_n > Y_\epsilon$ and $\lambda = y_n/z_n \in (1 - \epsilon, 1 + \epsilon)$ in (A.21) we obtain that for all $n > \max\{N_\epsilon, M_\epsilon\}$ we have

$$1 - 2\epsilon < \lambda - \epsilon < \left(\frac{f^{-1}(y_n)}{f^{-1}(z_n)}\right)^\alpha < \lambda + \epsilon < 1 + 2\epsilon.$$

As for all $n \in \mathbb{N}$ we have $f^{-1}(y_n) = g^{-1}(z_n)$, the preceding statement reads: for all $n > \max\{N_\epsilon, M_\epsilon\}$ we have

$$1 - 2\epsilon < \left(\frac{g^{-1}(z_n)}{f^{-1}(z_n)}\right)^\alpha < 1 + 2\epsilon.$$

Since $\epsilon \in \left(0, \frac{1}{2}\right]$ was arbitrary, the last displayed relationship proves that $g^{-1} \overset{s}{\sim} f^{-1}$ at $+\infty$ which is equivalent to $f^{-1} \overset{s}{\sim} g^{-1}$ at $+\infty$.

The converse follows when we apply the already proven implication to the regularly varying function $f^{-1}$ at $+\infty$ and an increasing unbounded function $g^{-1}$.

We prove (b) by change of variables. Assume that $f$ and $g$ are decreasing with $0$ as the limit and that $f$ is regularly varying at $+\infty$ with negative index $\alpha$. Define the functions $f_1(x) = 1/f(x)$ and $g_1(x) = 1/g(x)$. Then $f_1^{-1}(x) = f^{-1}(1/x)$ and $g_1^{-1}(x) = g^{-1}(1/x)$ and $f_1$ and $g_1$ are increasing and unbounded. Clearly, $f \overset{s}{\sim} g$ at $+\infty$ if and only if $f_1 \overset{s}{\sim} g_1$ at $+\infty$ and $f_1^{-1} \overset{s}{\sim} g_1^{-1}$ at $+\infty$ if and only if $f^{-1} \overset{s}{\sim} g^{-1}$ at $0_+$. Also, $f_1$ is regularly varying at $+\infty$ with positive index $-\alpha$. Hence, by part (a), $f_1 \overset{s}{\sim} g_1$ at $+\infty$ if and only if $f_1^{-1} \overset{s}{\sim} g_1^{-1}$ at $+\infty$. The last three equivalences prove (b). $\qquad\square$

The preceding theorem and the method utilized in the proof of its part (b) yield the next corollary.

**Corollary A.12.** *Let $f$ and $g$ be strictly monotonic positive functions defined in a neighbourhood of $0_+$ and let $f$ be regularly varying at $0_+$ with a nonzero index.*

(a) *If $f$ and $g$ are increasing with $0$ limit at $0_+$, then the inverses $f^{-1}$ and $g^{-1}$ are also increasing, defined in a neighbourhood of $0_+$ and the following equivalence holds*

$$f \overset{s}{\sim} g \text{ at } 0_+ \quad \Leftrightarrow \quad f^{-1} \overset{s}{\sim} g^{-1} \text{ at } 0_+.$$

(b) *If $f$ and $g$ are decreasing and unbounded, then the inverses $f^{-1}$ and $g^{-1}$ are decreasing, defined in a neighbourhood of $+\infty$ and the following equivalence holds*

$$f \overset{s}{\sim} g \text{ at } 0_+ \quad \Leftrightarrow \quad f^{-1} \overset{s}{\sim} g^{-1} \text{ at } +\infty.$$

The following corollary is a consequence of the fact that the negation of $f \overset{s}{\sim} g$ at $+\infty$ is the statement

$$\left(\frac{f(x)}{g(x)} - 1\right)^{-1} = O(1) \quad \text{as} \quad x \to +\infty.$$

The similar negation is easily formulated for $f \overset{s}{\sim} g$ at $0_+$. Each of the four statements in Theorem A.11 and Corollary A.12 can be expressed using one of these negations. We state only the analogue of the last statement in Corollary A.12 since that is what is used in Theorem 4.17.

**Corollary A.13.** *Let $f$ and $g$ be strictly monotonic positive functions defined in a neighbourhood of $0_+$ and let $f$ be regularly varying at $0_+$ with a nonzero index. If $f$ and $g$ are decreasing and unbounded, then the inverses $f^{-1}$ and $g^{-1}$ are decreasing, defined in a neighbourhood of $+\infty$ and the following equivalence holds*

$$\left(\frac{f(x)}{g(x)} - 1\right)^{-1} = O(1) \ as \ x \downarrow 0 \quad \Leftrightarrow \quad \left(\frac{f^{-1}(y)}{g^{-1}(y)} - 1\right)^{-1} = O(1) \ as \ y \to +\infty.$$

Clearly $f \sim g$ at $+\infty$ implies $f \overset{s}{\sim} g$ at $+\infty$. In the next example we will demonstrate that $f \overset{s}{\sim} g$ at $+\infty$ does not imply $f \sim g$ at $+\infty$ even for smooth normalized slowly varying increasing functions $f$ and $g$ for which $f/g$ is normalized slowly varying function.

**Example A.14.** Consider the following two increasing functions

$$\phi(u) = u + \sin u, \qquad \psi(u) = u - \sin u, \quad u \in \mathbb{R}_+.$$

Then, $\phi(u) - \psi(u) = 2\sin u$, and all three functions $\phi, \psi$ and $\phi - \psi$ satisfy the condition on the right-hand side of the equivalence in (A.4). Therefore the functions

$$f(x) = \exp\big(\ln_2 x + \sin(\ln_2 x)\big) \quad \text{and} \quad g(x) = \exp\big(\ln_2 x - \sin(\ln_2 x)\big),$$

as well as the ratio

$$\frac{f(x)}{g(x)} = \exp\big(2\sin(\ln_2 x)\big) = \exp\big((\phi - \psi)(\ln_2 x)\big),$$

are normalized slowly varying functions at $+\infty$. Notice that $f$ and $g$ are increasing and unbounded. For every $n \in \mathbb{N}$ set

$$x_n = \exp\big(\exp(n\pi)\big), \quad y_n = (x_{2n-1})^{\exp(-\pi/2)}, \quad z_n = (x_{2n-1})^{\exp(\pi/2)},$$

and calculate

$$\frac{f(x_n)}{g(x_n)} = 1, \quad \frac{f(y_n)}{g(y_n)} = e^2, \quad \frac{f(z_n)}{g(z_n)} = e^{-2}.$$

Hence $f \overset{s}{\sim} g$ at $+\infty$ holds and $f(x) \sim g(x)$ as $x \to +\infty$ is not true. To get a sense how fast the above sequences grow we list their second entries:

$$x_2 \approx 3.64 * 10^{232}, \quad y_2 \approx 5.37 * 10^{1118}, \quad z_2 \approx 1.52 * 10^{25888},$$

and point out that $x_{12}, y_8$, and $z_7$ are larger than the maximum number available to Wolfram Mathematica on a 64-bit Windows computer.

A.4. **Positively increasing functions.** The following class of functions was introduced as a generalization of regularly varying functions with positive index, see [13, Section 3.1 and Definition 3.26].

**Definition A.15.** Let $a \in \mathbb{R}_+$. A nondecreasing function $f : (0, a] \to \mathbb{R}_+$ is called *positively increasing at* 0 *from the right* if there exists $\lambda \in (0, 1)$ such that

$$\limsup_{x \downarrow 0} \frac{f(\lambda x)}{f(x)} < 1.$$

A function $g : [-a, 0) \to \mathbb{R}_-$ is called *positively increasing at* 0 *from the left* if the function $f(x) = -g(-x), x \in [-a, 0)$, is positively increasing at 0 from the right.

A function $g : [a, +\infty) \to \mathbb{R}_+$ is called *positively increasing at* $+\infty$ if the function $f(x) = 1/g(1/x), x \in (0, 1/a]$, is positively increasing at 0 from the right. A function $g : (-\infty, -a] \to \mathbb{R}_-$ is called *positively increasing at* $-\infty$ if the function $f(x) = -1/g(-1/x), x \in (0, 1/a]$, is positively increasing at 0 from the right.

The relationship between regularly varying and positively increasing functions at $+\infty$, and analogously at $-\infty$, $0_+$ and $0_-$, is as follows. Each regularly varying function with positive index is positively increasing, while a regularly varying function with a nonpositive index is not positively increasing. In particular, a slowly varying function is not positively increasing. The exponential function $\exp$ is positively increasing at $+\infty$ but not regularly varying at $+\infty$, see Figure 2.

The next proposition is analogous to Proposition A.3.

**Proposition A.16.** *Let $a \in \mathbb{R}_+$ and let $f : [a, +\infty) \to \mathbb{R}_+$ be a nondecreasing function. Set $b = \ln a$ and set $\phi$ to be the composition $\ln \circ f \circ \exp$ defined on $[b, +\infty)$. The function $f$ is positively increasing at $+\infty$ if and only if there exists $c \in \mathbb{R}_+$ such that*

$$\lim_{u \uparrow +\infty} \big(\phi(u) - \phi(u - c)\big) = +\infty \quad or \quad \liminf_{u \uparrow +\infty} \big(\phi(u) - \phi(u - c)\big) > 0.$$

*Proof.* The definition of $\phi$ implies that

$$f(x) = \exp\big(\phi(\ln x)\big) \quad \text{for all} \quad x \in [a, +\infty).$$

Therefore, for all $\lambda \in (0, 1)$ and all $x \in [a/\lambda, +\infty)$ we have

$$\frac{f(\lambda x)}{f(x)} = \exp\big(-\big(\phi(\ln x) - \phi(\ln x + \ln \lambda)\big)\big).$$

Now the claim follows from the continuity of the function $\exp(-x)$, the fact that the function $-\ln x$ is a bijection between $(0, 1)$ and $\mathbb{R}_+$ and the fact that $\ln x \to +\infty$ as $x \to +\infty$. $\qquad\square$

**Example A.17.** In this example we present a nondecreasing function $f : [1, +\infty) \to \mathbb{R}_+$ which is neither positively increasing nor slowly varying. Let $\phi : [0, +\infty)$ be a continuous piecewise linear function such that $\phi(0) = 0$ and for each $n \in \mathbb{N}$ on the closed interval $\big[n^2, n^2 + n\big]$ it has the value $n(n+1)/2$, see Figure 1. This function can be defined piecewise as follows: for every $n \in \mathbb{N}$ we have

$$\phi(x) = \frac{n(n+1)}{2} + \min\big\{0, x - n^2\big\} \quad \text{where} \quad x \in (n^2 - n, n^2 + n].$$

Since for every $n \in \mathbb{N}$ we have

$$\phi'(x) = 1 \quad \text{if} \quad x \in (n^2 - n, n^2) \quad \text{and} \quad \phi'(x) = 0 \quad \text{if} \quad x \in (n^2, n^2 + n),$$

the function $\phi$ is nondecreasing and by the fundamental theorem of calculus for $x_0, x \in \mathbb{R}_+$ such that $x_0 \leq x$ we have

$$x - \phi(x) - x_0 + \phi(x_0) = \int_{x_0}^{x} \big(1 - \phi'(\xi)\big) d\xi \geq 0.$$

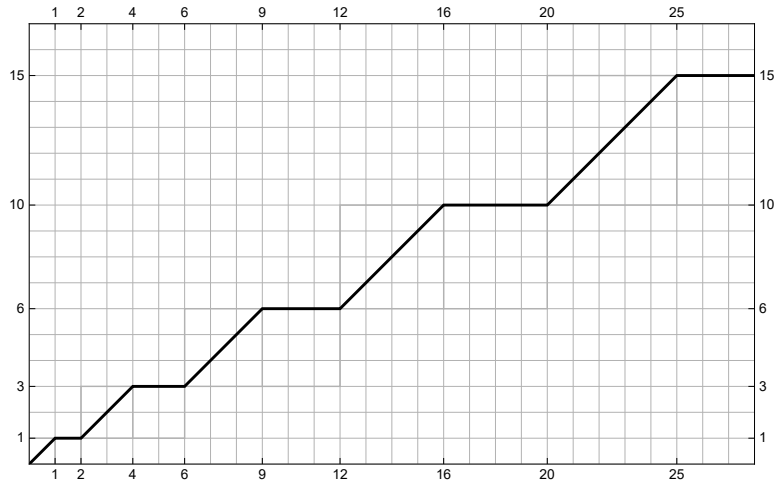That is, for an arbitrary $x_0 \in \mathbb{R}_+$ we have

$$\phi(x) \leq x - x_0 + \phi(x_0) \quad \text{for all} \quad x \in [x_0, +\infty). \tag{A.22}$$

Let $c \in \mathbb{R}_+$ be arbitrary and let $x \in \mathbb{R}_+$ be such that $x > c$. Then substituting $x_0 = x - c$ in (A.22) and using the fact that $\phi$ is nondecreasing we get

$$0 \leq \phi(x) - \phi(x - c) \leq c \quad \text{for all} \quad x \in [c, +\infty).$$

Further, for all $n \in \mathbb{N}$ such that $n > c$ we have

$$\phi\big(n^2\big) - \phi\big(n^2 - c\big) = \frac{1}{2}n(n+1) - \left(\frac{1}{2}n(n+1) - c\right) = c$$

FIGURE 1. The function $\phi$ in Example A.17

and
$$\phi\bigl(n^2 + n\bigr) - \phi\bigl(n^2 + n - c\bigr) = \frac{1}{2}n(n+1) - \frac{1}{2}n(n+1) = 0.$$
The last three displayed relations yield that for all $c \in \mathbb{R}_+$ we have
$$\liminf_{u\uparrow+\infty}\bigl(\phi(u) - \phi(u - c)\bigr) = 0 \quad \text{and} \quad \limsup_{u\uparrow+\infty}\bigl(\phi(u) - \phi(u - c)\bigr) = c > 0.$$
The preceding two equalities and Propositions A.3 and A.16 imply that the nondecreasing function
$$f(x) = \exp\bigl(\phi(\ln x)\bigr) \quad \text{where} \quad x \in [1, +\infty)$$
is neither slowly varying nor it is positively increasing.

We point out that the integer sequence $1, 2, 4, 6, 9, 12, 16, 20, \ldots$ of the values of the independent variable at which $\phi$ is not differentiable has been exhaustively studied in [72].

In Figure 2 we present a Venn diagram with the universal set of nondecreasing functions. On this diagram we informally present three classes of functions that are most relevant for this paper. The black dot marked by exp represent the exponential function, while the other black dot represents the function $f$ from Example A.17.

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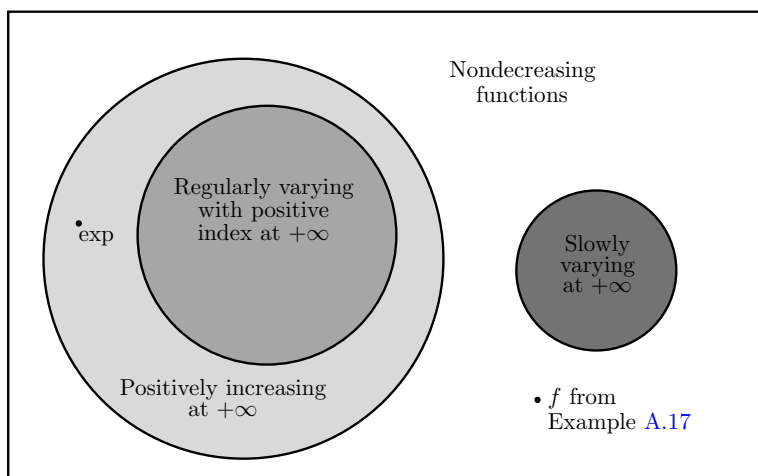
FIGURE 2. The universal set of nondecreasing functions

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