Linear Algebra and its Applications 679 (2023) 86–98

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Operators without eigenvalues in finite-dimensional vector spaces: Essential uniqueness of the model

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A R T I C L E   I N F O

Article history:
Received 26 July 2023
Received in revised form 31 August 2023
Accepted 2 September 2023
Available online 7 September 2023
Submitted by P. Semrl

MSC:
15A03
46C20
46E22
47A06
47A45

Keywords:
Matrix polynomial
Pontryagin space
Reproducing kernel
Symmetric operator
Boundary mapping
Nilpotent operator
Self-adjoint extension
Canonical space of vector polynomials

A B S T R A C T

In [4] a model is presented of a finite-dimensional Pontryagin space with a symmetric operator without eigenvalues. In this note we show that this model is unique up to an equivalence relation.

Published by Elsevier Inc.

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https://doi.org/10.1016/j.laa.2023.09.001
0024-3795/Published by Elsevier Inc.

1. Introduction

This note is an amendment to our paper [4], in which we presented a model for a finite-dimensional Pontryagin space $(\mathfrak{G}, [\,\cdot\,,\cdot\,]_{\mathfrak{G}})$ in which there is defined a symmetric operator S without eigenvalues. The model consists of a finite-dimensional vector space $\mathfrak{C}_\mu$ of d-vector polynomials and the operator S_μ in $\mathfrak{C}_\mu$ of multiplication by the independent variable; here $d = \operatorname{codim}(\operatorname{dom} S)$ and μ is a d-tuple of positive integers determined by S, see the main Theorem 2.4 in the next section. The space $\mathfrak{C}_\mu$ is a reproducing kernel space with reproducing kernel defined by an invertible self-adjoint matrix Q and a matrix polynomial $\mathcal{P}(z)$ both with additional properties which are formulated in Theorem 2.1. The purpose of this note is to show that this model is in one-to-one correspondence with the pair $\{\mathsf{Q}, \mathcal{P}(z)\}$ up to equivalence. This is what we mean by the essential uniqueness of the model in the title. In Theorem 2.3 we explain the equivalence relation.

In a sequel [5] to [4] and this note we apply our results to relate the resolvent set, the spectrum, eigenfunctions and Jordan chains of a self-adjoint extension (with finite-dimensional exit space) of a closed symmetric linear relation S in a Krein space to an eigenvalue problem for S^*, the adjoint of S, with boundary conditions that depend polynomially on the eigenvalue parameter.

We have tried to make this note self-contained, but to avoid repetition in the proofs, we refer, where possible, to proofs already given in [4].

By $\mathbb{C}^d[z]$, $d \in \mathbb{N}$, we denote the vector space of d-vector polynomials in z. Let μ be a d-tuple of positive integers: $\mu = (\mu_1, \dots, \mu_d)$ with $\mu_1 \geq \cdots \geq \mu_d \geq 1$. By $\mathfrak{C}_\mu$ we denote the subspace of $\mathbb{C}^d[z]$ of vector polynomials

$$\begin{bmatrix} p_1(z) \\ \vdots \\ p_d(z) \end{bmatrix} \quad \text{with} \quad \deg p_j(z) < \mu_j, \quad j \in \{1, \dots, d\}.$$

Such a space will be called a *canonical subspace* of $\mathbb{C}^d[z]$. The operator S_μ on $\mathfrak{C}_\mu$ of multiplication by z is an operator without eigenvalues and $d = \operatorname{codim}(\operatorname{dom} S_\mu)$. Moreover, S_μ is nilpotent. If m is the index of nilpotency and

$$\delta_j := \dim(\operatorname{dom} S_\mu^{j-1}), \quad j \in \{1, \dots, m+1\},$$

(so that $\delta_1 = \dim \mathfrak{C}_\mu$ and $\delta_{m+1} = 0$), then $\mu = (\mu_1, \dots, \mu_d)$ with

$$\mu_j = \#\bigl\{i \in \{1, \dots, m\} : \delta_i - \delta_{i+1} \geq j\bigr\}, \quad j \in \{1, \dots, d\},$$

where $\#\Omega$ stands for the number of elements in the set Ω. (In terms of notions related to Young diagrams, $\mu = \mathsf{Con\,Der}\,\delta$, where $\delta = (\delta_1, \dots, \delta_m)$, see [4, Section 5].)

The defect numbers of a closed symmetric operator S in a Pontryagin space $(\mathfrak{H}, [\,\cdot\,,\cdot\,]_{\mathfrak{H}})$ are by definition the defect numbers of the closed symmetric operator JS in the Hilbert space $\mathfrak{H}$ equipped with the positive definite J-inner product $[J\cdot\,,\cdot\,]_{\mathfrak{H}}$, where J is an arbitrary fundamental symmetry on $(\mathfrak{H}, [\,\cdot\,,\cdot\,]_{\mathfrak{H}})$. The defect numbers are equal if and only if S has a self-adjoint extension A in $(\mathfrak{H}, [\,\cdot\,,\cdot\,]_{\mathfrak{H}})$ and then their common value equals $\dim(A/S)$. Assume S has equal defect numbers d. Then an operator $\mathsf{b} : S^* \to \mathbb{C}^{2d}$, where S^* is the adjoint of S, is called a *boundary mapping for* S if b is linear, $\operatorname{ran} \mathsf{b} = \mathbb{C}^{2d}$ and $\ker \mathsf{b} = S$. Alternatively, a linear operator $\mathsf{b} : S^* \to \mathbb{C}^{2d}$ is a boundary mapping if there exists a $2d \times 2d$ matrix Q such that

$$[g, h]_{\mathfrak{H}} - [f, k]_{\mathfrak{H}} = \mathrm{i}\, \mathsf{b}(\{h, k\})^* \mathsf{Q} \mathsf{b}(\{f, g\}) \quad \text{for all} \quad \{f, g\}, \{h, k\} \in S^*. \tag{1.1}$$

Q is called the *Gram matrix for* b and uniquely determined by b. It is invertible, self-adjoint, and has d positive and d negative eigenvalues. If b and $\widehat{\mathsf{b}}$ are boundary mappings for S and have Gram matrices Q and $\widehat{\mathsf{Q}}$, then there is an invertible $2d \times 2d$ matrix U such that

$$\widehat{\mathsf{Q}} = U^* \mathsf{Q} U \quad \text{and} \quad \widehat{\mathsf{b}} = U^{-1} \mathsf{b}.$$

Indeed, applying (1.1) we obtain the equality

$$\mathsf{b}(\{h, k\})^* \mathsf{Q} \mathsf{b}(\{f, g\}) = \widehat{\mathsf{b}}(\{h, k\})^* \widehat{\mathsf{Q}} \widehat{\mathsf{b}}(\{f, g\}) \quad \text{for all} \quad \{f, g\}, \{h, k\} \in S^*. \tag{1.2}$$

Define the linear relation U in $\mathbb{C}^{2d} \times \mathbb{C}^{2d}$ by

$$U := \Big\{ \big\{ \widehat{\mathsf{b}}\big(\{f, g\}\big), \mathsf{b}\big(\{f, g\}\big) \big\} \, : \, \{f, g\} \in S^* \Big\}.$$

Since $\widehat{\mathsf{b}}$ and b are boundary mappings, $\operatorname{dom} U = \operatorname{ran} U = \mathbb{C}^{2d}$. If $\widehat{\mathsf{b}}\big(\{f, g\}\big) = 0$, then $\{f, g\} \in S$ and hence also $\mathsf{b}\big(\{f, g\}\big) = 0$. The converse also holds. Thus $\ker U = \ker U^{-1} = \{0\}$ and U is the graph of an invertible matrix, which we also denote by U:

$$U \widehat{\mathsf{b}}\big(\{f, g\}\big) = \mathsf{b}\big(\{f, g\}\big), \quad \{f, g\} \in S^*.$$

If we substitute this in (1.2), we obtain $\widehat{\mathsf{Q}} = U^* \mathsf{Q} U$.

In the next sections, we use the following conventions. For vector functions $a(z)$ and $b(z)$, the identity $a(z) \equiv b(z)$ stands for the proposition $a(z) = b(z)$ for all $z \in \mathbb{C}$. For $m, n \in \mathbb{N}$ we denote by $\mathbb{C}^{m \times n}[z]$ the space of all matrix polynomials with coefficients in $\mathbb{C}^{m \times n}$. The *degree of such a polynomial* is $-\infty$ if it is the zero polynomial, otherwise it is the highest power of z for which the corresponding matrix coefficient is nonzero. A square matrix polynomial is called *unimodular* if its determinant is a nonzero scalar. As already done above we write $\mathbb{C}^m[z]$ for $\mathbb{C}^{m \times 1}[z]$. For $z \in \mathbb{C}$ by z^* we denote the complex conjugate of z. Also, * denotes the complex conjugate of a matrix.

2. Uniqueness of the model

The first theorem in this section gives sufficient conditions on a matrix Q and a matrix polynomial $\mathcal{P}(z)$ under which the reproducing kernel space with reproducing kernel defined by (2.1) below is a canonical subspace $\mathfrak{C}_\mu$ of $\mathbb{C}^d[z]$ and describes–and we think this is new–the adjoint of and a boundary mapping for the symmetric operator S_μ of multiplication by the independent variable z acting on this space. For the formula of the adjoint in this theorem, see also [2, Theorem 6.2]; in the proof we repeat some arguments from [3, p. 1327].

Theorem 2.1. *Let $d \in \mathbb{N}$, let Q be an invertible self-adjoint $2d \times 2d$ matrix with d positive and d negative eigenvalues and let $\mathcal{P}(z)$ be a $d \times 2d$ matrix polynomial whose j-th row has degree μ_j with $j \in \{1, \dots, d\}$. Assume that $\mathcal{P}(z)$ has the following properties:*

(a) $\mathcal{P}(z)\mathsf{Q}^{-1}\mathcal{P}(z^*)^* = 0$ *for all* $z \in \mathbb{C}$.
(b) $\operatorname{rank} \mathcal{P}(z) = d$ *for all* $z \in \mathbb{C}$.
(c) $\operatorname{rank} \mathcal{P}_\infty = d$, *where* $\mathcal{P}_\infty := \lim_{z\to\infty} \operatorname{diag}(z^{-\mu_1}, \dots, z^{-\mu_d})\mathcal{P}(z)$.
(d) $\mu_1 \geq \cdots \geq \mu_d \geq 1$.

Then:

(i) *The Pontryagin space $\mathfrak{K}_\mathcal{P}$ with reproducing kernel*

$$K_\mathcal{P}(z, w) = \frac{\mathrm{i}}{z - w^*}\,\mathcal{P}(z)\mathsf{Q}^{-1}\mathcal{P}(w)^*, \quad z \neq w^*, \quad z, w \in \mathbb{C}, \tag{2.1}$$

is the canonical subspace $\mathfrak{C}_\mu$ of $\mathbb{C}^d[z]$ where $\mu = (\mu_1, \dots, \mu_d)$.

(ii) *The operator S_μ of multiplication by the independent variable is symmetric in this space, its defect numbers are equal to d and its adjoint is given by*

$$S_\mu^* = \big\{\{f, g\} \in \mathfrak{C}_\mu^2 \,:\, zf(z) - g(z) \equiv \mathcal{P}(z)c \text{ for some } c \in \mathbb{C}^{2d}\big\}.$$

(iii) *The linear relation*

$$\Big\{\{\{f, g\}, c\} \in \mathfrak{C}_\mu^2 \times \mathbb{C}^{2d} \,:\, \{f, g\} \in S_\mu^* \text{ and } zf(z) - g(z) \equiv \mathcal{P}(z)c\Big\}$$

is the graph of an operator $\mathsf{b}_\mu : S_\mu^* \to \mathbb{C}^{2d}$ which is a boundary mapping for S_μ with Gram matrix Q.

The inner product in $\mathfrak{C}_\mu$ relative to which S_μ is symmetric is determined by the kernel (2.1). To emphasize this we sometimes write $\mathfrak{C}_\mu = \mathfrak{K}_\mathcal{P}$ or $\mathfrak{C}_\mu = (\mathfrak{C}_\mu, K_\mathcal{P})$.

Remark 2.2. In the next section, see the last sentence there, we show that the properties (a)-(d) of $\mathcal{P}(z)$ imply that

$$\bigcap_{z \in \mathbb{C}} \ker \mathcal{P}(z) = \{0\}. \tag{2.2}$$

This equality does not play a role in [4, Sections 10 and 11], but it is now a frequently used tool in the proofs that follow.

Proof of Theorem 2.1. (i) This statement is proved in [4, Theorem 10.3].

(ii) Step 1. By [3, Theorem 1.1 and its proof], S_μ is symmetric. We prove that its defect numbers are both equal to d. Notice that $d = \operatorname{codim}(\operatorname{dom} S_\mu)$ and that S_μ is an operator without eigenvalues. Let J be a fundamental symmetry on $\mathfrak{C}_\mu$. Then according to [4, Lemma 3.5] JS_μ has a self-adjoint *operator* extension B in $\mathfrak{C}_\mu$ equipped with the J-inner product, which implies that S_μ has equal defect numbers and $A := JB$ is a self-adjoint extension of S_μ in $\mathfrak{C}_\mu$ which is also an operator. Thus $\operatorname{dom} A$ is dense in $\mathfrak{G}$, in fact $\operatorname{dom} A = \mathfrak{C}_\mu$, since $\mathfrak{C}_\mu$ is finite dimensional. Thus the defect numbers of S_μ are equal to

$$\dim(A/S_\mu) = \dim\big((\operatorname{dom} A)/(\operatorname{dom} S_\mu)\big) = \dim\big(\mathfrak{C}_\mu/(\operatorname{dom} S_\mu)\big) = \operatorname{codim}(\operatorname{dom} S_\mu) = d.$$

Step 2. By [2, Remark 3.5 and Theorem 3.7], a generalized Von Neumann's formula holds for S_μ:

$$S_\mu^* = S_\mu + S_\mu^* \cap w_0 I + S_\mu^* \cap w_0^* I + \sum_{j=1}^{\kappa} S_\mu^* \cap w_j I,$$

where κ is the negative index of the Pontryagin space $\mathfrak{C}_\mu$ and the w_j's belong to $\mathbb{C} \setminus \mathbb{R}$ and satisfy $w_j \neq w_k^*$ for all $j, k \in \{0, \dots, \kappa\}$. Set

$$\mathcal{L}_{w^*} = \big\{\{K_{\mathcal{P}}(\,\cdot\,, w)x, w^* K_{\mathcal{P}}(\,\cdot\,, w)x\} \,:\, x \in \mathbb{C}^d\big\}$$

and

$$\mathcal{L} = \operatorname{span}\{\mathcal{L}_{w^*} \,:\, w \in \mathbb{C}\}.$$

We show that

$$S_\mu^* \cap w^* I = \mathcal{L}_{w^*} \quad \text{for all} \quad w \in \mathbb{C}. \tag{2.3}$$

Let $w \in \mathbb{C}$. First, we establish that both spaces in (2.3) have dimension d. That $\dim(S_\mu^* \cap w^* I) = d$ follows from Lemma [4, Lemma 3.4]. To show that $\dim \mathcal{L}_{w^*} = d$, assume that for some $x \in \mathbb{C}^d$ we have $K_{\mathcal{P}}(\,\cdot\,, w)x = 0$. This means that $\mathcal{P}(z)\mathsf{Q}^{-1}\mathcal{P}(w)^* x = 0$ for all $z \in \mathbb{C}$. By equality (2.2), we have $\mathcal{P}(w)^* x = 0$, and then from property (b) we obtain that $x = 0$. Hence $\dim \mathcal{L}_{w^*} = d$. Since both spaces have the same dimension, to prove that they coincide, it suffices to prove the inclusion $\mathcal{L}_{w^*} \subseteq S_\mu^* \cap w^* I$. Let $x \in \mathbb{C}^d$ and $\{K_{\mathcal{P}}(\,\cdot\,, w)x,\, w^* K_{\mathcal{P}}(\,\cdot\,, w)x\} \in \mathcal{L}_{w^*}$. Then, by the reproducing property of the kernel $K_{\mathcal{P}}$, we have that for all $\{f, g\} \in S_\mu$

$$\big[g, K_{\mathcal{P}}(\,\cdot\,, w)x\big]_{\mathfrak{K}_{\mathcal{P}}} - \big[f, w^* K_{\mathcal{P}}(\,\cdot\,, w)x\big]_{\mathfrak{K}_{\mathcal{P}}} = \big\langle g(w), x\big\rangle_{\mathbb{C}^d} - \big\langle w f(w), x\big\rangle_{\mathbb{C}^d} = 0.$$

Hence $\{K_{\mathcal{P}}(\,\cdot\,,w)x,\, w^*K_{\mathcal{P}}(\,\cdot\,,w)x\} \in S_\mu^* \cap w^*I$. This proves the desired inclusion and equality (2.3). From (2.3) and the generalized Von Neumann's formula we obtain

$$S_\mu^* = S_\mu + \mathcal{L}. \tag{2.4}$$

Step 3. Using (2.4), we prove the equality in the theorem. Denote its right-hand side by T_μ. By its definition, $S_\mu \subseteq T_\mu$. Let $x \in \mathbb{C}^d$, $w \in \mathbb{C}$ and let $\{K_{\mathcal{P}}(\,\cdot\,,w)x, w^*K_{\mathcal{P}}(\,\cdot\,,w)x\} \in \mathcal{L}_{w^*}$. Then

$$zK_{\mathcal{P}}(z,w)x - w^*K_{\mathcal{P}}(z,w)x = \mathcal{P}(z)c \quad \text{with} \quad c = \mathrm{i}\,\mathsf{Q}^{-1}\mathcal{P}(w)^*x.$$

Hence $\mathcal{L}_{w^*} \in T_\mu$ for all $w \in \mathbb{C}$. Consequently, $S_\mu + \mathcal{L} \subseteq T_\mu$. To prove the reverse inclusion, let $\{f,g\} \in T_\mu$. Then $\{f,g\} \in \mathfrak{C}_\mu^2$ and $zf(z) - g(z) \equiv \mathcal{P}(z)c$ for some $c \in \mathbb{C}^{2d}$. By equality (2.2)

$$\operatorname{span}\{\operatorname{ran}\mathcal{P}(w)^* \,:\, w \in \mathbb{C}\} = \mathbb{C}^{2d},$$

and hence c can be written as a finite sum of the form

$$c = \mathrm{i}\,\mathsf{Q}^{-1}\sum_w \mathcal{P}(w)^*c_w \quad \text{with} \quad c_w \in \mathbb{C}^d. \tag{2.5}$$

Define

$$\{f_1,g_1\} = \sum_w \{K_{\mathcal{P}}(\,\cdot\,,w)c_w, w^*K_{\mathcal{P}}(\,\cdot\,,w)c_w\}. \tag{2.6}$$

Then, by (2.4), $\{f_1,g_1\} \in \mathcal{L}$. Using (2.1), we find that $\{f,g\} - \{f_1,g_1\} \in S_\mu$. It follows that $\{f,g\} \in S_\mu + \mathcal{L}$. Thus we have shown that $T_\mu \subseteq S_\mu + \mathcal{L}$. Consequently, $S_\mu^* = T_\mu$.

(iii) If $\{\{f,g\},c\} \in \mathfrak{C}_\mu^2 \times \mathbb{C}^{2d}$ and $\{f,g\} = 0$, then $\mathcal{P}(z)c = 0$ for all $z \in \mathbb{C}$. From equality (2.2), it follows that $c = 0$. Hence, the linear relation is an operator which, as in the theorem, we denote by b_μ. By item (ii), $\operatorname{dom}\mathsf{b}_\mu = S_\mu^*$, $\operatorname{ran}\mathsf{b}_\mu \subseteq \mathbb{C}^{2d}$ and $\ker\mathsf{b}_\mu = S_\mu$. We show that b_μ is surjective. Let $c \in \mathbb{C}^{2d}$ and let $\{f_1,g_1\} \in S_\mu^*$ be defined as in the proof of item (ii), see (2.5) and (2.6). Then

$$zf_1(z) - g_1(z) \equiv \mathcal{P}(z)c,$$

that is, $\mathsf{b}_\mu(\{f_1,g_1\}) = c$.

Finally, we prove that Q is the Gram matrix for b_μ. Let $\{f,g\},\{h,k\} \in S_\mu^*$ and set $c = \mathsf{b}_\mu(\{f,g\})$ and $a = \mathsf{b}_\mu(\{h,k\})$. Define $\{f_1,g_1\}$ as above. Similarly, with a written as the finite sum

$$a = \mathrm{i}\,\mathsf{Q}^{-1}\sum_v \mathcal{P}(v)^*a_v \quad \text{with} \quad a_v \in \mathbb{C}^d,$$

define

$$\{h_1, k_1\} = \sum_v \big\{K_{\mathcal{P}}(\,\cdot\,, v)a_v, v^* K_{\mathcal{P}}(\,\cdot\,, v)a_v\big\}.$$

Then $\{f_1, g_1\}$ and $\{h_1, k_1\}$ belong to S_μ^* and the differences

$$\{f_1, g_1\} - \{f, g\} \quad \text{and} \quad \{h_1, k_1\} - \{h, k\}$$

belong to S_μ. This and the reproducing property of the kernel $K_{\mathcal{P}}$ imply

$$\begin{aligned}
[g, h]_{\mathfrak{K}_{\mathcal{P}}} - [f, k]_{\mathfrak{K}_{\mathcal{P}}} &= [g_1, h_1]_{\mathfrak{K}_{\mathcal{P}}} - [f_1, k_1]_{\mathfrak{K}_{\mathcal{P}}} \\
&= \sum_{v,w} (v - w^*)\big\langle K_{\mathcal{P}}(v, w)c_w, a_v\big\rangle_{\mathbb{C}^d} \\
&= \mathrm{i} \sum_{v,w} a_v^* \mathcal{P}(v)\mathsf{Q}^{-1}\mathcal{P}(w)^* c_w \\
&= \mathrm{i}\, a^* \mathsf{Q} c \\
&= \mathrm{i}\, \mathsf{b}_\mu(\{h, k\})^* \mathsf{Q} \mathsf{b}_\mu(\{f, g\}).
\end{aligned}$$

This shows that b_μ is a boundary mapping for S_μ with Gram matrix Q. □

Denote by $\mathfrak{S}$ the set of all pairs $\{\mathsf{Q}, \mathcal{P}(z)\}$ in which Q is a self-adjoint invertible $2d \times 2d$ matrix with d positive and d negative eigenvalues and $\mathcal{P}(z)$ is a $d \times 2d$ matrix polynomial with the properties (a)-(d) of Theorem 2.1. We say the pairs $\{\mathsf{Q}, \mathcal{P}(z)\}$ and $\{\widehat{\mathsf{Q}}, \widehat{\mathcal{P}}(z)\}$ in $\mathfrak{S}$ are related and denote it by $\{\mathsf{Q}, \mathcal{P}(z)\} \sim \{\widehat{\mathsf{Q}}, \widehat{\mathcal{P}}(z)\}$, if there exists a unimodular $d \times d$ matrix polynomial $\mathcal{W}(z) = \big[w_{jk}(z)\big]_{j,k=1}^d$ satisfying

$$\deg w_{jk}(z) \le \mu_j - \mu_k \quad \text{for all} \quad j, k \in \{1, \dots, d\}, \tag{2.7}$$

such that

$$\widehat{\mathcal{P}}(z)\widehat{\mathsf{Q}}^{-1}\widehat{\mathcal{P}}(w)^* = \mathcal{W}(z)\mathcal{P}(z)\mathsf{Q}^{-1}\mathcal{P}(w)^*\mathcal{W}(w)^* \quad \text{for all} \quad z, w \in \mathbb{C}. \tag{2.8}$$

Theorem 2.3.

(a) *The relation $\sim$ is an equivalence relation on $\mathfrak{S}$.*

(b) *For $\{\mathsf{Q}, \mathcal{P}(z)\}$ and $\{\widehat{\mathsf{Q}}, \widehat{\mathcal{P}}(z)\} \in \mathfrak{S}$ the following statements are equivalent:*

- (i) $\{\mathsf{Q}, \mathcal{P}(z)\} \sim \{\widehat{\mathsf{Q}}, \widehat{\mathcal{P}}(z)\}$.
- (ii) *There exists an isomorphism $\Psi : (\mathfrak{C}_\mu, K_{\mathcal{P}}) \to (\mathfrak{C}_\mu, K_{\widehat{\mathcal{P}}})$ satisfying $\Psi S_\mu = S_\mu \Psi$.*
- (iii) *There exist an invertible $2d \times 2d$ matrix V and a unimodular $d \times d$ matrix polynomial $\mathcal{W}(z) = \big[w_{jk}(z)\big]_{j,k=1}^d$ satisfying (2.7) such that*

$$\widehat{\mathsf{Q}} = V^* \mathsf{Q} V \quad \textit{and} \quad \widehat{\mathcal{P}}(z) \equiv \mathcal{W}(z)\mathcal{P}(z)V. \tag{2.9}$$

(c) *If* (i)-(iii) *in* (b) *hold then* Ψ *is the operator of multiplication by* $\mathcal{W}(z)$ *and the boundary mappings* b_μ *and* $\widehat{\mathsf{b}}_\mu$ *associated with* S_μ *in the spaces* $(\mathfrak{C}_\mu, K_{\mathcal{P}})$ *and* $(\mathfrak{C}_\mu, K_{\widehat{\mathcal{P}}})$ *satisfy the relation*

$$\widehat{\mathsf{b}}_\mu\big(\{\mathcal{W}f, \mathcal{W}g\}\big) = V^{-1}\mathsf{b}_\mu(\{f,g\}) \quad \textit{for all} \quad \{f,g\} \in S_\mu^*,$$

where $\mathcal{W}f$ *stands for the polynomial* $\mathcal{W}(z)f(z)$*.*

Proof. To prove item (a) we use [4, Theorem 6.2], which implies that there exists a bijection $\Phi : (\mathfrak{C}_\mu, K_{\mathcal{P}}) \to (\mathfrak{C}_\mu, K_{\widehat{\mathcal{P}}})$ satisfying $\Phi S_\mu = S_\mu \Phi$ if and only if Φ is the operator of multiplication by a unimodular $d\times d$ matrix polynomial $\mathcal{W}(z)$ satisfying (2.7). It follows that if $\mathcal{W}_1(z)$ and $\mathcal{W}_2(z)$ are unimodular $d \times d$ matrix polynomials satisfying (2.7), then so are the inverse $\mathcal{W}_1(z)^{-1}$ and the product $\mathcal{W}_1(z)\mathcal{W}_2(z)$. From this, it readily follows that the relation $\sim$ is reflexive, symmetric and transitive, hence an equivalence relation.

We now prove item (b). Assume (i). Then, by (2.8),

$$\mathcal{W}(z)K_{\mathcal{P}}(z,w)\mathcal{W}(w)^* = K_{\widehat{\mathcal{P}}}(z,w).$$

Since $\ker \mathcal{W}(\cdot) = \{0\}$ and by [1, Theorem 1.5.7], the operator of multiplication by $\mathcal{W}(z)$ is an isomorphism from $(\mathfrak{C}_\mu, K_{\mathcal{P}})$ onto $(\mathfrak{C}_\mu, K_{\widehat{\mathcal{P}}})$. Clearly, it intertwines the operators S_μ. This proves (ii). Now assume (ii). We prove (iii) together with item (c). Since Ψ is a bijection that intertwines the multiplication operators by z and by [4, Theorem 6.2], there exists a matrix $\mathcal{W}(z)$ with the properties asserted in the theorem such that $(\Psi f)(z) = \mathcal{W}(z)f(z)$, $f(z) \in \mathfrak{C}_\mu$. We denote by $S_\mu^\times$ the adjoint of S_μ in the space $(\mathfrak{C}_\mu, K_{\widehat{\mathcal{P}}})$. Since Ψ is an isomorphism,

$$S_\mu^\times = \Psi S_\mu^* \Psi^{-1}$$

or in full

$$\begin{aligned} S_\mu^\times &= \big\{\{\Psi f, \Psi g\} \,:\, \{f,g\} \in \mathfrak{C}_\mu^2,\ zf(z) - g(z) \equiv \mathcal{P}(z)\mathsf{b}_\mu(\{f,g\})\big\} \\ &= \big\{\{u,v\} \,:\, \{u,v\} \in \mathfrak{C}_\mu^2,\ zu(z) - v(z) \equiv \mathcal{W}(z)\mathcal{P}(z)\mathsf{b}_\mu(\{\Psi^{-1}u, \Psi^{-1}v\})\big\}. \end{aligned}$$

On the other hand

$$S_\mu^\times = \big\{\{u,v\} \,:\, \{u,v\} \in \mathfrak{C}_\mu^2,\ zu(z) - v(z) \equiv \widehat{\mathcal{P}}(z)\widehat{\mathsf{b}}_\mu(\{u,v\})\big\}.$$

Now consider the relation V in $\mathbb{C}^{2d}$ defined by

$$V := \big\{\{\widehat{\mathsf{b}}_\mu(\{u,v\}), \mathsf{b}_\mu(\{\Psi^{-1}u, \Psi^{-1}v\})\} \,:\, \{u,v\} \in S_\mu^\times\big\}.$$

Since $\widehat{\mathsf{b}}_\mu$ is a boundary mapping, $\operatorname{dom} V = \mathbb{C}^{2d}$. Since $\Psi^{-1}S_\mu^\times\Psi = S_\mu^*$ and b_μ is a boundary mapping, $\operatorname{ran} V = \mathbb{C}^{2d}$. We show that V is an operator. Indeed, if $\widehat{\mathsf{b}}_\mu(\{u,v\}) = 0$ for some $\{u, v\} \in S_\mu^\times$, then $\{u, v\} \in S_\mu$ and hence, since $\Psi^{-1} S_\mu \Psi = S_\mu$, $\{\Psi^{-1}u, \Psi^{-1}v\} \in S_\mu$ which shows that $\mathsf{b}_\mu(\{\Psi^{-1}u, \Psi^{-1}v\}) = 0$. The converse also holds. Thus V defines an invertible $2d \times 2d$ matrix and

$$\mathsf{b}_\mu\big(\{\Psi^{-1}u, \Psi^{-1}v\}\big) = V\widehat{\mathsf{b}}_\mu(\{u, v\}) \quad \text{for all} \quad \{u, v\} \in S_\mu^\times.$$

This equality and the two formulas for $S_\mu^\times$ imply the last two equalities in the theorem. It remains to prove $\widehat{\mathsf{Q}} = V^*\mathsf{Q}V$. Since the isomorphism Ψ amounts to multiplication by $\mathcal{W}(z)$ it follows that the canonical subspace $\big(\mathfrak{C}_\mu, K_{\widehat{\mathcal{P}}}\big)$ has two reproducing kernels which necessarily coincide:

$$\mathcal{W}(z)K_{\mathcal{P}}(z, w)\mathcal{W}(w)^* = K_{\widehat{\mathcal{P}}}(z, w),$$

see [1, Theorem 1.5.7]. By writing this out, using the equality $\widehat{\mathcal{P}}(z) \equiv \mathcal{W}(z)\mathcal{P}(z)V$ and the fact that $\mathcal{W}(z)$ is invertible, we find that

$$\mathcal{P}(z)\big(\mathsf{Q}^{-1} - V\widehat{\mathsf{Q}}^{-1}V^*\big)\mathcal{P}(w)^* = 0 \quad \text{for all} \quad z, w \in \mathbb{C}.$$

Applying (2.2) we find that

$$\big(\mathsf{Q}^{-1} - V\widehat{\mathsf{Q}}^{-1}V^*\big)\mathcal{P}(w)^* = 0 \quad \text{for all} \quad w \in \mathbb{C}.$$

Since (2.2) also implies that $\operatorname{span}\{\operatorname{ran} P(w)^* : w \in \mathbb{C}\} = \mathbb{C}^{2d}$, it follows that

$$\mathsf{Q}^{-1} = V\widehat{\mathsf{Q}}^{-1}V^*.$$

From this, we obtain the asserted equality. Finally the equalities for $\widehat{\mathsf{Q}}$ and $\widehat{\mathcal{P}}(z)$ in (2.9) imply the equality (2.8), that is, (iii) implies (i). □

The next theorem, the main theorem, provides a model for a finite-dimensional Pontryagin space on which there is defined a symmetric operator without eigenvalues. The model is a canonical subspace $\mathfrak{C}_\mu$ together with the operator S_μ uniquely determined by an equivalence class in the set $\mathfrak{S}$. The model appeared already in [4], but the essential uniqueness is new.

Theorem 2.4. *Let $(\mathfrak{G}, [\,\cdot\,,\cdot\,]_{\mathfrak{G}})$ be a finite-dimensional Pontryagin space and let S be a symmetric operator on $\mathfrak{G}$ without eigenvalues. Then the defect numbers of S are equal to $d := \operatorname{codim}(\operatorname{dom} S)$ and S is nilpotent. With m being the nilpotency index of S and*

$$\delta_j := \dim(\operatorname{dom} S^{j-1}), \quad j \in \{1, \ldots, m+1\}, \quad (\textit{so that } \delta_1 = \dim \mathfrak{G},\ \delta_{m+1} = 0),$$

let $\mu = (\mu_1, \mu_2, \ldots, \mu_d)$ be the d-tuple with entries

$$\mu_j = \#\big\{i \in \{1, \ldots, m\} : \delta_i - \delta_{i+1} \geq j\big\}, \quad j \in \{1, \ldots, d\}.$$

Then there exist

(i) *a pair* $\{\mathsf{Q}, \mathcal{P}(z)\} \in \mathfrak{S}$ *with* $\mu_j = \deg \mathcal{P}_j(z)$, *where* $\mathcal{P}_j(z)$ *is the* j*-th row of* $\mathcal{P}(z)$,
(ii) *an isomorphism* $\Phi : \mathfrak{G} \to (\mathfrak{C}_\mu, K_{\mathcal{P}})$ *with* $\Phi S = S_\mu \Phi$, *where the canonical subspace* $\mathfrak{C}_\mu$, *the multiplication operator* S_μ *and the kernel* $K_{\mathcal{P}}$ *are related to* $\{\mathsf{Q}, \mathcal{P}(z)\}$ *as in Theorem* 2.1.

The pair $\{\mathsf{Q}, \mathcal{P}(z)\}$ *is unique up to equivalence in* $\mathfrak{S}$.

Proof. That S is nilpotent follows from [4, Proposition 3.6]. A proof based on [4, Lemma 3.5] that S has defect numbers both equal to d can be given similar to the proof that S_μ in Theorem 2.1 has defect numbers equal to d. The remaining statements in items (i) and (ii) are established by [4, Theorem 11.1]. It remains to prove the essential uniqueness. Assume $\{\widehat{\mathsf{Q}}, \widehat{\mathcal{P}}(z)\}$ is a pair in $\mathfrak{S}$ for which there exists an isomorphism $\widehat{\Phi} : \mathfrak{G} \to (\mathfrak{C}_\mu, K_{\widehat{\mathcal{P}}})$. Then $\Psi := \widehat{\Phi}\Phi^{-1} : (\mathfrak{C}_\mu, K_{\mathcal{P}}) \to (\mathfrak{C}_\mu, K_{\widehat{\mathcal{P}}})$ is an isomorphism satisfying with $\Psi S_\mu = S_\mu \Psi$. Hence, by Theorem 2.3 (b) we have $\{\mathsf{Q}, \mathcal{P}(z)\} \sim \{\widehat{\mathsf{Q}}, \widehat{\mathcal{P}}(z)\}$. $\square$

3. On the kernel of $\mathcal{P}(z)$

In this section, we prove Remark 2.2. We assume $d \in \mathbb{N}$ and $\mathcal{P}(z)$ is a nonzero $d \times 2d$ matrix polynomial with rows $\mathcal{P}_j(z)$:

$$\mathcal{P}(z) = \begin{bmatrix} \mathcal{P}_1(z) \\ \vdots \\ \mathcal{P}_d(z) \end{bmatrix} \quad \text{where} \quad \mathcal{P}_j(z) \in \mathbb{C}^{1\times 2d}[z].$$

We set $\sigma_j = \deg \mathcal{P}_j(z) \in \{0\} \cup \mathbb{N}$, $j \in \{1, \ldots, d\}$, and define the $d \times 2d$ matrix

$$\mathcal{P}_\infty := \lim_{z\to\infty} \operatorname{diag}(z^{-\sigma_1}, \ldots, z^{-\sigma_d})\mathcal{P}(z).$$

We denote by p the degree of $\mathcal{P}(z)$. Thus $p \in \{0\} \cup \mathbb{N}$. Expanding $\mathcal{P}(z)$ in powers of z:

$$\mathcal{P}(z) = P_0 + P_1 z + \cdots + P_p z^p, \quad P_j \in \mathbb{C}^{d\times 2d},$$

we associate with $\mathcal{P}(z)$ the $(p+1)d \times 2d$ *coefficient matrix* P

$$P := \begin{bmatrix} P_0 \\ P_1 \\ \vdots \\ P_p \end{bmatrix}.$$

Then

$$\mathcal{P}(z) = [\, I_d \quad zI_d \quad \cdots \quad z^p I_d \,]\, P, \quad z \in \mathbb{C}, \tag{3.1}$$

and this readily implies that

$$\ker P = \bigcap_{z\in\mathbb{C}} \ker \mathcal{P}(z) = \bigcap_{z\in\mathbb{W}} \ker \mathcal{P}(z), \tag{3.2}$$

whenever $\mathbb{W} \subset \mathbb{C}$ and $\#\mathbb{W} \geq p+1$. Indeed, the set on the left is contained in the set in the middle and the set in the middle is contained in the set on the right. To prove that the set on the right is contained in the set on the left, let $\mathbb{W}$ be a subset of $\mathbb{C}$ containing mutually different elements $w_0, w_1, \ldots, w_p$ and define the invertible Vandermonde $(p+1)d\times(p+1)d$ matrix M by

$$M = \begin{bmatrix} I_d & w_0 I_d & \cdots & w_0^p I_d \\ I_d & w_1 I_d & \cdots & w_1^p I_d \\ \vdots & \vdots & \ddots & \vdots \\ I_d & w_p I_d & \cdots & w_p^p I_d \end{bmatrix}.$$

Then $\mathcal{P}(w_k)x = 0$ for all $k \in \{0,\ldots,p\}$ implies, by (3.1), $MPx = 0$ and hence $Px = 0$, proving (3.2).

We say that $\mathcal{P}(z)$ has the *predictable degree property* if for every $1\times d$ vector polynomial $u(z) = \begin{bmatrix} u_1(z) & \cdots & u_d(z) \end{bmatrix}$ we have

$$\deg\bigl(u(z)\mathcal{P}(z)\bigr) = \max\bigl\{\sigma_j + \deg u_j(z), \quad j \in \{1,\ldots,d\}\bigr\}. \tag{3.3}$$

The following lemma is the key to the proof of Theorem 3.2 below. It is well-known in system theory, specifically in convolutional coding. For proofs see [10, Theorem A.2], [8, Theorems 2.22 and 2.28], [9, Theorem 6.3-13] and Forney's fundamental paper [6].

Lemma 3.1. *If* $\operatorname{rank}\mathcal{P}(z) = d$ *for some* $z \in \mathbb{C}$*, then* $\operatorname{rank}\mathcal{P}_\infty = d$ *if and only if* $\mathcal{P}(z)$ *has the predictable degree property.*

Theorem 3.2. *Let* Q *be an invertible self-adjoint* $2d \times 2d$ *matrix with* d *positive and* d *negative eigenvalues. If* $\mathcal{P}(z)$ *has the properties*

(a) $\mathcal{P}(z)\mathsf{Q}^{-1}\mathcal{P}(z^*)^* = 0$ *for all* $z \in \mathbb{C}$*,*
(b) $\operatorname{rank}\mathcal{P}(z) = d$ *for all* $z \in \mathbb{C}$*,*
(c) $\operatorname{rank}\mathcal{P}_\infty = d$*,*

then

$$\dim\Bigl(\bigcap_{z\in\mathbb{W}} \ker \mathcal{P}(z)\Bigr) = \#\bigl\{j \in \{1,\ldots,d\} \,:\, \sigma_j = 0\bigr\} \tag{3.4}$$

for any subset $\mathbb{W}$ *of* $\mathbb{C}$ *with* $\#\mathbb{W} \geq p+1$.

Proof. In view of (3.2) it suffices to show that

$$\dim \ker P = \sum_{j=1}^{d} \max\{0, 1-\sigma_j\}.$$

We denote by $\boldsymbol{\pi} : \mathbb{C}^{2d}[z] \to \mathbb{C}^{d}[z]$ and by $\boldsymbol{\tau} : \mathbb{C}^{d}[z] \to \mathbb{C}^{2d}[z]$ the operators of multiplication by $\mathcal{P}(z)$ and $\mathcal{T}(z) := \mathsf{Q}^{-1}\mathcal{P}(z^*)^*$. By (b), $\mathcal{T}(z)$ has full rank for all $z \in \mathbb{C}$. Therefore the Smith Normal Form theorem (see for example [7, Satz 6.3] or [9, Theorem 6.3-16]) yields the existence of a unimodular $d \times d$ matrix polynomial $\mathcal{E}(z)$ and a unimodular $2d \times 2d$ matrix polynomial $\mathcal{F}(z)$ such that

$$\mathcal{T}(z) = \mathcal{F}(z) \begin{bmatrix} I_d \\ 0 \end{bmatrix} \mathcal{E}(z) \quad \text{for all} \quad z \in \mathbb{C}. \tag{3.5}$$

This implies that $\boldsymbol{\tau}$ is injective.

It follows from (3.1) that P is the matrix representation of the restriction $\boldsymbol{\pi}|_{\mathbb{C}^{2d}}$ of $\boldsymbol{\pi}$ to $\mathbb{C}^{2d}$ with respect to the standard bases in $\mathbb{C}^{2d}$ and the subspace of polynomials in $\mathbb{C}^{d}[z]$ of degree $< p+1$. It follows that

$$\ker P = (\ker \boldsymbol{\pi}) \cap \mathbb{C}^{2d}. \tag{3.6}$$

By (a) we have $\boldsymbol{\pi} \circ \boldsymbol{\tau} = 0$, hence $\operatorname{ran} \boldsymbol{\tau} \subseteq \ker \boldsymbol{\pi}$. We show that, in fact, equality holds. For that, consider $f(z) \in \ker \boldsymbol{\pi}$. Then $\mathcal{P}(z)f(z) \equiv 0$ and, by (a) and (b), for every $z \in \mathbb{C}$ there exists a unique $c_z \in \mathbb{C}^d$ such that $f(z) = \mathcal{T}(z)c_z$. From (3.5) it follows that

$$c_z = \mathcal{E}(z)^{-1} \begin{bmatrix} I_d & 0 \end{bmatrix} \mathcal{F}(z)^{-1} f(z),$$

hence c_z is a polynomial in $\mathbb{C}^d[z]$ and $f(z) \in \operatorname{ran} \boldsymbol{\tau}$. This proves the asserted equality. It follows that

$$(\ker \boldsymbol{\pi}) \cap \mathbb{C}^{2d} = (\operatorname{ran} \boldsymbol{\tau}) \cap \mathbb{C}^{2d}. \tag{3.7}$$

Now (3.6), (3.7) and the injectivity of $\boldsymbol{\tau}$ imply the following chain of equalities

$$\begin{aligned}
\dim \ker P &= \dim((\operatorname{ran} \boldsymbol{\tau}) \cap \mathbb{C}^{2d}) \\
&= \dim \boldsymbol{\tau}^{-1}(\mathbb{C}^{2d}) \\
&= \dim\{f(z) \in \mathbb{C}^d[z] \,:\, \mathcal{T}(z)f(z) \in \mathbb{C}^{2d}\} \\
&= \dim\{f(z) \in \mathbb{C}^d[z] \,:\, \deg(\mathcal{P}(z^*)^* f(z)) < 1\} \\
&= \dim\{u(z) \in \mathbb{C}^{1\times d}[z] \,:\, \deg(u(z)\mathcal{P}(z)) < 1\} \\
&= \sum_{j=1}^{d} \max\{0, 1-\sigma_j\}.
\end{aligned}$$

The last equality follows from (c) and Lemma 3.1. Indeed, using the notation as in (3.3) we obtain

$$\begin{aligned} \deg(u(z)\mathcal{P}(z)) < 1 \quad &\Leftrightarrow \quad \sigma_j + \deg u_j(z) < 1 \quad \text{for all} \quad j \in \{1, \dots, d\} \\ &\Leftrightarrow \quad \deg u_j(z) < 1 - \sigma_j \quad \text{for all} \quad j \in \{1, \dots, d\}. \end{aligned}$$

This proves (3.4). □

In the previous section we have that $\deg \mathcal{P}_j(z) = \sigma_j = \mu_j \geq 1$, for all $j \in \{1, \dots, d\}$, and hence (2.2) in Remark 2.2 follows from (3.4) in Theorem 3.2.

Declaration of competing interest

The authors have declared that no competing interest exists.

Data availability

No data was used for the research described in the article.

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