AN EXCEPTIONAL EXPONENTIAL FUNCTION

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There are at least two surprising results in this article. The first is that there is actually a link between the standard calculus problem of finding the best view of a painting and graphs of exponential functions. The second is that the exponential function with the "best view" is not the one with the base e. As a bonus we get an interesting appearance of the Lambert W function.

The painting problem

One of the standard calculus problems is this:

> A painting hangs on a wall in an art gallery. Its top edge is at distance p and its bottom edge at distance q above the eye level of an art lover. How far from the wall should the art lover stand to get the best view? (Here the "best view" means the maximum viewing angle; see Figure 1.)

This statement of the painting problem is similar to Exercise 44 in Stewart [9, p. 314]. In Problem 23 of Hughes-Hallett, et al. [6, p. 203], a painting is replaced by the Statue of Liberty and the values of p and q are specific numbers. In this setting the requirement that the bottom of the observed object be above the eye level of an observer seems more natural.

We leave the details of the solution of the painting problem to the reader. Before proceeding, however, we make an observation. For each distance from the gallery wall, there is a corresponding viewing angle. Moreover, for each distance v other than $\sqrt{qp}$ from the wall there is a second distance qp/v with the same viewing angle. Figure 1 gives a geometric proof of this fact.

Ray $\overrightarrow{OD}$ in the figure represents eye level. Let $OA = OA' = p$, $OB = OB' = q$, $OC = OC' = v$, and $OD = qp/v$. Right triangles $\triangle OAC$ and $\triangle OA'C'$ are congruent since their legs are congruent. Similarly, right triangles $\triangle OBC$ and $\triangle OB'C'$ are congruent. Hence the marked angles at C and C' are congruent. Furthermore, right triangles $\triangle OA'C'$

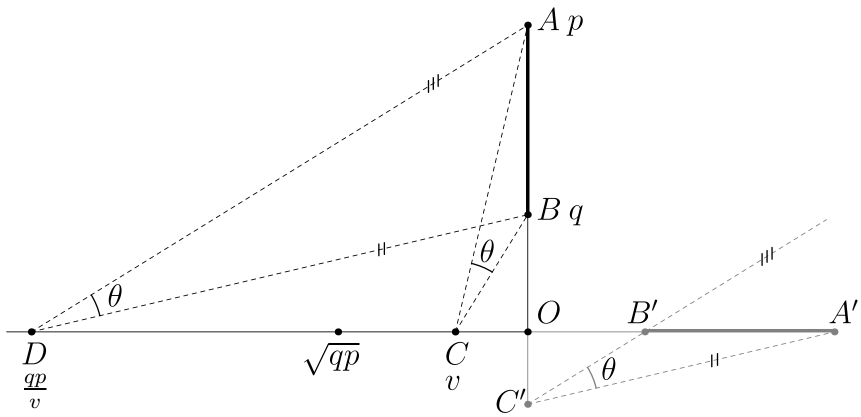

Figure 1. A painting on a wall.

and $\triangle ODB$ are similar since their legs are proportional:

$$\frac{OD}{OA'} = \frac{qp/v}{p} = \frac{q}{v}, \qquad \frac{OB}{OC'} = \frac{q}{v}.$$

Therefore lines $\overleftrightarrow{C'A'}$ and $\overleftrightarrow{DB}$ are parallel. In the same way, triangles $\triangle OB'C'$ and $\triangle ODA$ are similar, which implies that lines $\overleftrightarrow{C'B'}$ and $\overleftrightarrow{DA}$ are also parallel. Hence the marked angles at C' and D are congruent. Consequently, the viewing positions C and D provide the same viewing angle θ.

If we assume that the solution of the painting problem is unique (that is, if we assume that there exists a viewing position from which the viewing angle is larger than that from any other position) then, by what we just proved, the distance $\sqrt{qp}$ is the only possible candidate for the solution. Using either calculus combined with trigonometry, or plane geometry, it can be shown that $\sqrt{qp}$ is indeed the unique solution. The calculus solution is standard. The plane geometry solution and much more on the painting problem can be found in [3].

We now proceed to a seemingly unrelated topic of two special tangent lines to an exponential function. But, the painting problem is coming back. Stay alert!

An exceptional exponential function

Let $a > 1$. Consider the exponential function with base a, $f_a(x) = a^x$, and its graph in the xy-plane. There are two distinguished points in this setting: the origin and the y-intercept. Therefore two noteworthy tangent lines to the graph of f_a are the one that passes through the origin and the one at the y-intercept. Their equations are

$$\ell_{0,a}(x) = (e \ln a)\, x \quad \text{ and } \quad \ell_{1,a}(x) = (\ln a)\, x + 1.$$

Since $a^{1/(\ln a)} = \left(e^{\ln a}\right)^{1/(\ln a)} = e$, the tangent line $\ell_{0,a}$ touches the graph of f_a at the point $\big(1/(\ln a), e\big)$. Notice that the second coordinate of this point does not depend on a. This may not be so surprising to the readers of this *Journal*, since this property of exponential functions appeared in [8].

For large a, the lines $\ell_{0,a}$ and $\ell_{1,a}$ are almost vertical, so the angle between them is small. For a close to 1, these lines are almost horizontal, so again the angle between them is small. This suggests the existence of a special base β that maximizes the angle between the two lines. What is this base?

The lines $\ell_{0,a}$ and $\ell_{1,a}$ intersect at the point

$$V_a = \left(\frac{1}{(e-1)\ln a}, \frac{e}{e-1}\right). \tag{1}$$

Surprisingly, the second coordinate of V_a does not depend on a. This fact provides the link between the family of exponential functions and the painting problem that we considered earlier. We can think of V_a as the eye of an art lover looking at the painting that is positioned between the points $(0, 0)$ and $(0, 1)$. The line $y = e/(e-1)$ represents eye level. The angle between the tangent lines $\ell_{0,a}$ and $\ell_{1,a}$ is the viewing angle. This is somewhat upside-down viewing, but it is nevertheless useful for obtaining the maximum angle. (See Figure 2.) In the notation of the painting problem, we have

$$p = \frac{e}{e-1} \quad \text{and} \quad q = \frac{1}{e-1}. \tag{2}$$

Using the fact that the best view is at a distance of $\sqrt{qp}$, we find that the maximum angle between our two lines is attained when they intersect at the point

$$\left(\frac{\sqrt{e}}{e-1}, \frac{e}{e-1}\right).$$

Comparing this with (1), we conclude that $\ln\beta = 1/\sqrt{e}$. Hence the special base is

$$\beta = e^{1/\sqrt{e}} \approx 1.834057.$$

This is the base for the exceptional exponential function in the title. Its graph is the heavy curve in Figure 2. The connection with the painting problem is the reason why we call $y = e^{x/\sqrt{e}}$ the *best-view* exponential function.

Next we use our result for the painting problem that says that the distances v and qp/v from the wall provide the same viewing angle, to

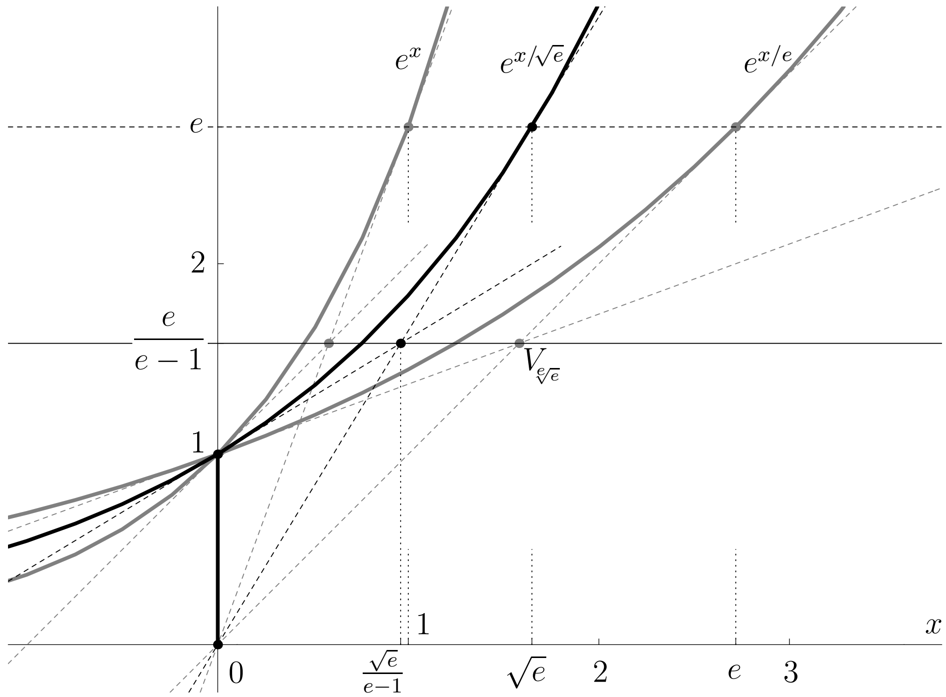

Figure 2. The exceptional exponential function: $e^{x/\sqrt{e}}$.

obtain the analogous result for exponential functions. Let

$$v = \frac{1}{(e-1)\ln a}$$

be the x-coordinate of V_a; see (1). With p and q from (2), we find that

$$\frac{qp}{v} = \frac{e\,\ln a}{e-1}.$$

Therefore the angle between the tangent lines intersecting at the point

$$\left(\frac{e\,\ln a}{e-1}, \frac{e}{e-1}\right)$$

is equal to the angle between the tangent lines intersecting at the point V_a. Comparing this with (1), we find that the corresponding base is $e^{1/(e\,\ln a)}$. Consequently, the angle between the noteworthy tangent lines to the graph of $y = a^x$ is equal to the angle between those tangent lines to the graph of $y = e^{x/(e\,\ln a)}$. Figure 2 shows this when $a = e$.

The plot of the radian measure of the angle between the lines $\ell_{0,a}$ and $\ell_{1,a}$ as a function of the base a is shown in Figure 3. The marked points represent the exponential functions plotted in Figure 2. Notice

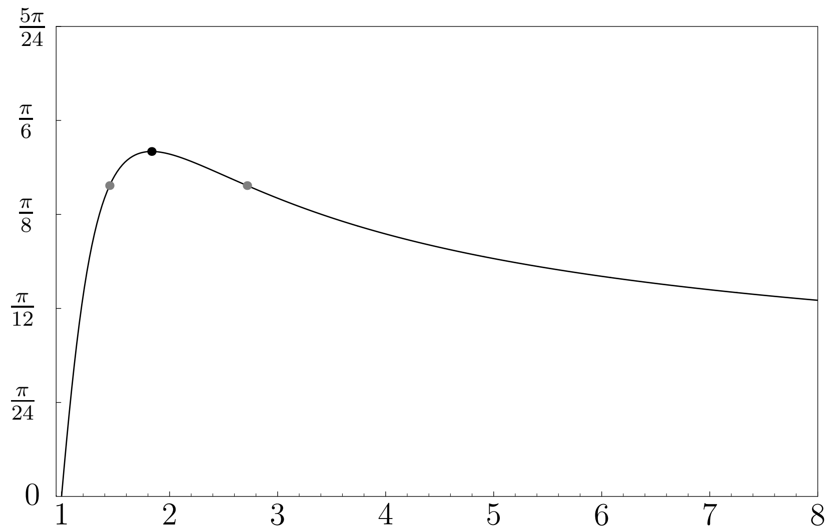

Figure 3. The angle between tangent lines as a function of base.

that the maximum angle (which corresponds to the base $e^{1/\sqrt{e}}$) is

$$\arctan\left(\frac{e-1}{2\sqrt{e}}\right) \approx 0.480\,\text{radians} \approx 27.524^\circ. \tag{3}$$

The angle that corresponds to the base e is

$$\arctan\left(\frac{e-1}{e+1}\right) \approx 0.433\,\text{radians} \approx 24.802^\circ.$$

Compositions of exponential functions and a linear function

In the preceding section we discovered the link between the family of exponential functions and the painting problem with $p = e/(e-1)$ and $q = 1/(e-1)$. Our next goal is to extend this result to families of functions obtained by composing exponential functions with a fixed linear function. We begin with a definition.

For $b > -1$ and $c > 0$, let $\mathcal{F}_{b,c}$ be the family of functions

$$g_a(x) = c\left(a^x + b\right) \quad \text{for} \quad a > 1.$$

For example, the family $\mathcal{F}_{0,1}$ is the family of exponential functions studied in the preceding section. Notice that $g_a = \varphi \circ f_a$, where $f_a(x) = a^x$ and $\varphi(x) = c(x+b)$.

Next we study two noteworthy tangent lines to the graphs of functions in $\mathcal{F}_{b,c}$. It is important to notice that the functions in $\mathcal{F}_{b,c}$ have the same y-intercept $\bigl(0, c(1+b)\bigr)$ and that $c(1+b) > 0$. One noteworthy tangent line is easy to find: The tangent line to the graph of g_a at the y-intercept is

$$\ell_{1,a}(x) = x\,c\,\ln a + c(1+b).$$

The existence and uniqueness of the other tangent line is not obvious; that is established by the following theorem.

Theorem. *Each function g_a in $\mathcal{F}_{b,c}$ has exactly one tangent line $\ell_{0,a}$ that passes through the origin and is tangent to the graph of g_a in the first quadrant.*

In the proof of this theorem we encounter an equation that does not seem to be solvable symbolically. However, with the help of *Mathematica* the author realized that the solution can be obtained in terms of the Lambert W function. In *Mathematica* it is the `ProductLog` function. For more about this function, which has attracted considerable attention recently, see for example [1, 2, 5, 7]. Before we begin the proof, we give definitions of W and its close relative Y, both of which play an important role in this article.

The functions W and Y and the proof

The art of solving equations is practiced with a large toolbox of inverse functions. When we encounter a function that is not one-to-one, we wisely restrict its domain, so that the resulting new function has an inverse. That is what is done here with the function $x \mapsto xe^x$. Although, this is a very elegant function, it is not one-to-one. But, a simple exercise in calculus shows that its restriction

$$T(x) = xe^x, \quad x \geq -1,$$

is strictly increasing. Therefore, T is a bijection from $[-1,+\infty)$ to $[-1/e,+\infty)$. We define W to be the inverse of T. Hence, W is a strictly increasing bijection from $[-1/e,+\infty)$ to $[-1,+\infty)$. The graphs of W and T are shown in Figure 4.

The function Y is defined by

$$Y(z) = \exp\bigl(1 + W(z/e)\bigr), \quad z \geq -1.$$

It is a composition of two strictly increasing bijections. Therefore Y is a strictly increasing bijection from $[-1,+\infty)$ to $[1,+\infty)$. An easy calculation shows the inverse of Y to be

$$Y^{-1}(x) = x\left(\ln x - 1\right), \quad x \geq 1.$$

Figure 5 shows Y and its inverse. Note that since $W(0) = 0$, $Y(0) = e$.

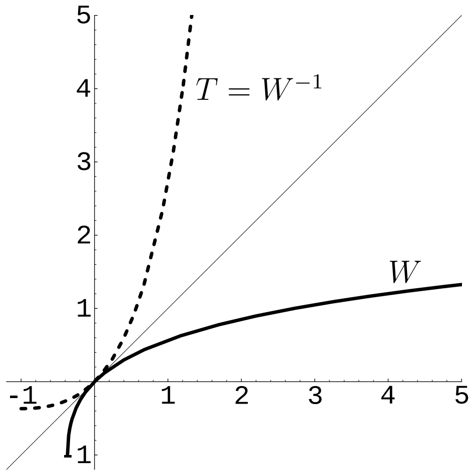

Figure 4. W and its inverse.

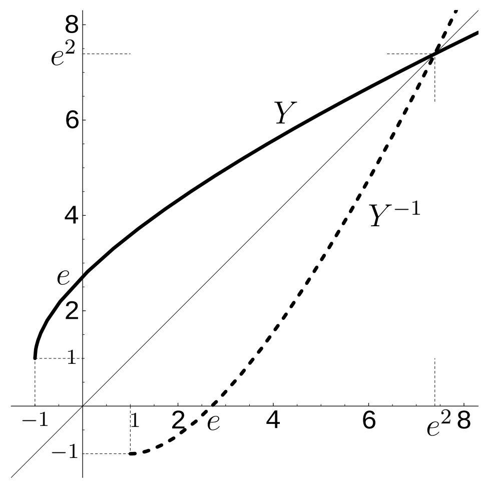

Figure 5. Y and its inverse.

The following relation between W and Y is a consequence of the definitions. For $z \geq -1$,

$$\begin{aligned} W(z/e)Y(z) &= W(z/e)\, \exp\bigl(1 + W(z/e)\bigr) \\ &= e\, W(z/e) \exp\bigl(W(z/e)\bigr) \\ &= e\, T\bigl(W(z/e)\bigr) \\ &= z. \end{aligned}$$

Therefore,

$$1 + W(z/e) = 1 + \frac{z}{Y(z)} = \frac{Y(z) + z}{Y(z)}, \quad \text{for} \quad z \geq -1. \tag{4}$$

Now we have all the tools we need to prove the theorem.

Proof. Let $a > 1$ and let $g_a(x) = c(a^x + b)$ be a function in $\mathcal{F}_{b,c}$. Further, let $\bigl(x_0, g_a(x_0)\bigr)$ be an arbitrary point on the graph of g_a in the first quadrant. The equation of the tangent line at this point is

$$\ell(x) = a^{x_0} c\,(\ln a)(x - x_0) + c\,\bigl(a^{x_0} + b\bigr). \tag{5}$$

A little computation shows that this line passes through the origin if and only if

$$\bigl(x_0 \ln a - 1\bigr) e^{x_0 \ln a - 1} = \frac{b}{e};$$

that is, if and only if

$$T\bigl(x_0 \ln a - 1\bigr) = \frac{b}{e}.$$

Since $b > -1$, the last equation has a unique solution. This is where the Lambert W function comes to the rescue. From the definition of

W and (4), we find that the only positive solution of the last equation is

$$x_0 = \frac{1 + W\big(b/e\big)}{\ln a} = \frac{Y(b) + b}{Y(b)\ \ln a}.$$

Substituting this into (5) gives the tangent line that passes through the origin. Since $a^{x_0} = e^{x_0 \ln a} = e^{1+W(b/e)} = Y(b)$, the equation of this line is

$$\ell_{0,a}(x) = x\, c\, Y(b)\ \ln a,$$

which completes the proof of the theorem. □

The painting problem appears again

We are now ready to look for a link between the painting problem and the graphs of the functions g_a in $\mathcal{F}_{b,c}$. Straightforward calculation shows that the intersection of $\ell_{1,a}$ and $\ell_{0,a}$ is

$$\left(\frac{1+b}{\big(Y(b)-1\big) \ln a}, \frac{(1+b)\, Y(b)\, c}{Y(b) - 1} \right). \tag{6}$$

Note that once again the second coordinate of this point does not depend on g_a. As before, we can think of this point as the eye of an art lover looking at the painting positioned between the points $(0,0)$ and $\big(0, c(1+b)\big)$. This is the link between the family $\mathcal{F}_{b,c}$ and the painting problem with

$$p = \frac{c(1+b)}{Y(b)-1}\, Y(b) \quad \text{and} \quad q = \frac{c(1+b)}{Y(b)-1}. \tag{7}$$

We shall, of course, illustrate this link with a picture (Figure 6), but before that we solve the inverse problem. (Note that (2) is the special case of (7) with $b = 0$ and $c = 1$.)

The inverse problem

In the previous section we saw that each family $\mathcal{F}_{b,c}$ is linked to a painting problem. Now we pose the inverse problem: Given a painting problem with parameters p and q, find the corresponding family $\mathcal{F}_{b,c}$.

To solve this problem we need to find b and c for which (7) holds. Our first attempt was to use *Mathematica*'s `Solve` command to solve (7) as a system of equations with unknowns b and c. Surprisingly, *Mathematica 4.1* did not solve this system. Nevertheless, the following simple reasoning leads to the solution. Let

$$F = \big\{(b,c)\ :\ b > -1,\ c > 0\big\} \quad \text{and} \quad G = \big\{(p,q)\ :\ p > q > 0\big\}.$$

First notice that the mapping from F to G given by (7) can be represented as the composition of the following three mappings:

$$\begin{aligned}(b,c) &\mapsto \big(b, c(1+b)\big) = (s,t),\\ (s,t) &\mapsto \big(Y(s), t\big) = (x,y),\\ (x,y) &\mapsto \left(\frac{xy}{x-1}, \frac{y}{x-1}\right) = (p,q).\end{aligned}$$

The reader can verify that the first mapping is a bijection with both domain and range F. The second mapping is a bijection between F and the set $\big\{(x,y) : x>1,\ y>0\big\}$. The third is a bijection between this set and G. Thus, the mapping given by (7), being the composition of three bijections, is a bijection from F to G. The inverses of the three bijections displayed above, listed in the reverse order, are

$$\begin{aligned}(p,q) &\mapsto \left(\frac{p}{q}, p-q\right) = (x,y),\\ (x,y) &\mapsto \big(Y^{-1}(x), y\big) = (s,t),\\ (s,t) &\mapsto \left(s, \frac{t}{1+s}\right) = (b,c).\end{aligned}$$

Composing these, starting with the first, we get

$$b = Y^{-1}\big(p/q\big) = \frac{p}{q}\ln\!\left(\frac{p}{qe}\right) \qquad \text{and} \qquad c = \frac{p-q}{1+b}. \tag{8}$$

This is the solution of the inverse problem: For a painting problem with parameters p and q, the family $\mathcal{F}_{b,c}$ with b and c given by (8) is linked to the painting problem.

For example, we used specific p and q from Figure 1 and formulas (8) to calculate the corresponding family $\mathcal{F}_{b,c}$ linked to the painting problem in Figure 1:

$$g_a(x) = \frac{p-q}{1+Y^{-1}\big(p/q\big)}\left(a^x + \frac{p}{q}\ln\!\left(\frac{p}{qe}\right)\right), \quad x\in\mathbb{R}, \qquad a>1.$$

The base a_C that corresponds to the point C in Figure 1 is found to be, using (6) and (8),

$$a_C = \exp\!\left(\frac{v\big(q+p\ln(p/qe)\big)}{p-q}\right).$$

The base a_D that corresponds to the point D is calculated similarly. This is illustrated in Figure 6. Notice that the main objects of Figure 1 appear upside down in Figure 6. The points A and B from Figure 1 are at the origin and the y-intercept in Figure 6. Notice that, for the viewing position C and the corresponding function g_{a_C}, the view lines through C are the noteworthy tangent lines to the graph of g_{a_C}. The heavy curve in Figure 6 is the graph of the best-view function. We calculate this function in the next section.

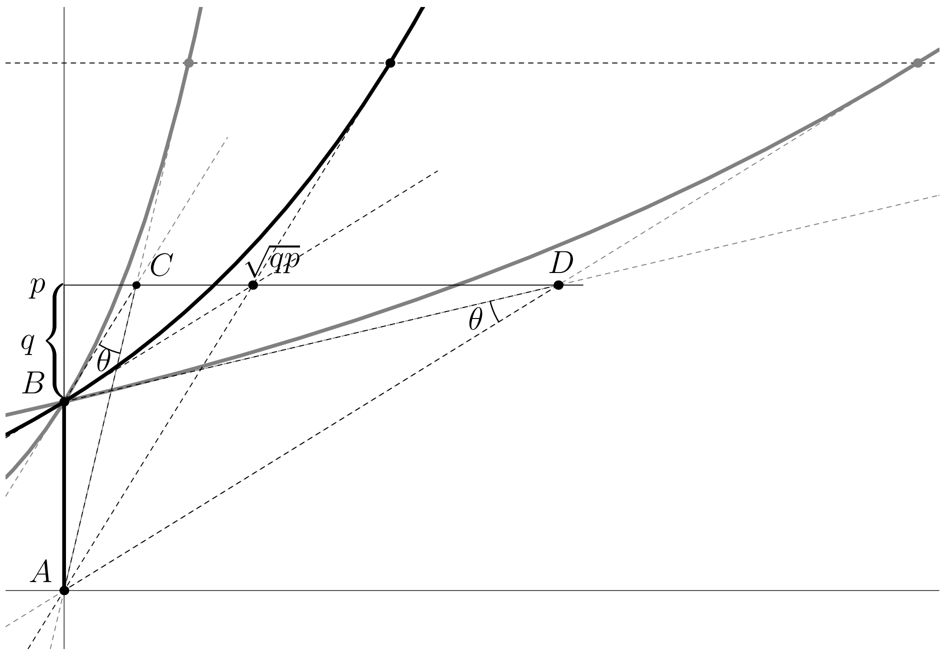

Figure 6. Figure 1 with the corresponding family $\mathcal{F}_{b,c}$. It is easier to recognize Figure 1 if the journal is turned upside down.

A family of exceptional functions

In each family $\mathcal{F}_{b,c}$ there is a special function g_β for which the noteworthy tangent lines form the maximum angle. Because of the connection with the painting problem, we call g_β the *best-view* function. Once more, recall that in the painting problem the best view is from distance $\sqrt{qp}$. Therefore, using (7), we find that the angle between the noteworthy tangent lines is maximum when they intersect at the point

$$\left(\frac{c(1+b)}{Y(b)-1}\, \sqrt{Y(b)}\, , \frac{1+b}{Y(b)-1}\, Y(b)\, c \right).$$

Comparing this point to the one in (6), we conclude that

$$\beta = e^{1/(c\sqrt{Y(b)})}.$$

Consequently, the best-view function in $\mathcal{F}_{b,c}$ is

$$g_\beta(x) = c\left(e^{x/(c\sqrt{Y(b)})} + b\right). \tag{9}$$

For $b > -1$ and $c > 0$, this is the family of exceptional functions in the title of this section. For $b = 0$ and $c = 1$, the exceptional exponential function is $y = e^{x/\sqrt{e}}$.

Where to hang a painting?

Assume that a painting of height 1 is to be placed in a gallery as described in the painting problem. How high should it be placed?

We give two ways to answer this question. It is common to associate the golden ratio ϕ with aesthetically pleasing balance between two lengths. Therefore, we first suggest choosing p and $q = p - 1$ so that both $p/1$ and $1/q$ equal the golden ratio. That is,

$$q = \frac{1}{\phi} = \frac{-1 + \sqrt{5}}{2} \approx 0.618 \quad \text{and} \quad p = \phi = \frac{1 + \sqrt{5}}{2} \approx 1.618.$$

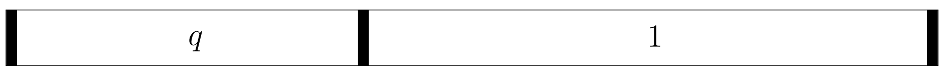

Figure 7. The golden ratio.

With this choice, the best view of the painting is at distance $\sqrt{qp} = 1$ from the wall. The corresponding viewing angle is

$$\arctan\left(\frac{1}{2}\right) \approx 0.464 \text{ radians} \approx 26.565^\circ.$$

Another way to answer the question is to let the exponential functions decide and so use the values for p and $q = p - 1$ suggested by the family $\mathcal{F}_{0,1}$:

$$q = \frac{1}{e - 1} \approx 0.582 \quad \text{and} \quad p = \frac{e}{e - 1} \approx 1.582.$$

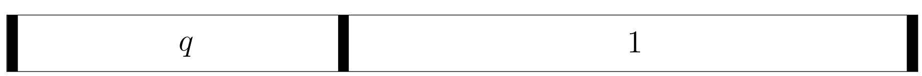

Figure 8. The "golden exponential ratio."

These numbers are remarkably close to the numbers suggested by the golden ratio, but they provide a slightly larger viewing angle of about 27.524°; see (3). (The percentage difference in p is only 2.228% and the percentage difference in the angle is 3.609%.) Can this be just a coincidence?

We invite the reader to use Figures 7 and 8 to make an esthetic comparison between the golden ratio and the "golden exponential ratio."

Closing comments

1. Let $g_a \in \mathcal{F}_{b,c}$ and $\ell_{0,a}$ be as in the theorem. It follows from the proof that their graphs meet at the point

$$\left(\frac{Y(b)+b}{Y(b)\,\ln a}, \bigl(Y(b)+b\bigr)\,c \right),$$

the second coordinate of which does not depend on the function itself. In the special case where $b=0$ and $c=1$, this property was proposed in [8] as a discover-e activity in a calculus class. For $b>-1$ and $c>0$ in general, this becomes a discover-W activity in a class that has been introduced to W as a new elementary function, as suggested in [1]. Inspired by [8], in [4] the authors gave a method for producing families of curves whose tangent lines at a fixed y-coordinate go through the origin. The family $\mathcal{F}_{b,c}$, however, cannot be obtained by this method.

2. At the beginning of this article we expressed our surprise that the exceptional exponential function was not the one with the base e. This expressed our particular affinity for the base e. Since we also have an affinity for the number 1, we pose the following problem: Find an exceptional function g in (9) whose y-intercept is $(0,1)$ and which is a composition of the exponential function e^x and a linear function.

3. As noted above, each function in $\mathcal{F}_{b,c}$ is the composition of an exponential function and a fixed linear function. This motivates the following problem: Find a non-linear function h for which the family $\mathcal{F}_h = \bigl\{ h \circ g : g \in \mathcal{F}_{0,1} \bigr\}$ has properties similar to $\mathcal{F}_{b,c}$. Some obvious candidates for h are quadratic functions and exponential functions.

4. For $a>1$ we considered graphs of exponential functions $f_a(x)=a^x$ and their tangent lines $\ell_{0,a}$ at the origin and $\ell_{1,a}$ at the y-intercept. These are special cases of the following problem.

Problem. Given an arbitrary point $(0,u)$ on the y-axis, find the equation of the tangent line to the graph of f_a that passes through $(0,u)$.

Although this problem looks like a simple exercise in calculus, its solution requires the use of the Lambert W function. In fact, the solution exists if and only if $u \le 1$. If a solution does exist, then the only two cases for u that can be found without the use of the Lambert function are $u=0$ and $u=1$. This is another reason why $\ell_{0,a}$ and $\ell_{1,a}$ are noteworthy.

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