EXISTENCE AND UNIQUENESS OF A JORDAN BASIS VIA YOUNG DIAGRAMS

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Abstract. For an operator $T$ on a finite-dimensional vector space over complex numbers we give a characterization of a Jordan basis and prove its existence mostly by counting on Young (also called Ferrers) diagrams related to $T$. We also give a model for $T$ in a space of vector polynomials.

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1. Introduction

In this note we present a proof of the existence and uniqueness of a Jordan basis for an operator on a finite-dimensional vector space $\mathfrak{F}$ over $\mathbb{C}$, see Theorems 6.1 and 6.5. Our proof is primarily based on counting with a Young diagram.

A Young diagram is an array of squares in which each subsequent row has no more squares than the row above it, as in the picture on the right. Equivalently, in a Young diagram each subsequent column has no more squares than the preceding column. By a tuple we mean a finite sequence. A Young diagram represents and is uniquely determined by the nonincreasing tuple of the numbers of squares in each row. In turn, this tuple uniquely determines the nonincreasing tuple of the numbers of squares in each of the columns of the corresponding Young diagram. These two tuples are said to be conjugate tuples. In the Young diagram shown on the right the conjugate tuples are $(5, 5, 4, 3, 3, 1)$ and $(7, 6, 6, 3, 2)$. Since counting with a Young diagram is the core of our proof, in Section 2 we give a rigorous presentation of the facts that we use. A Young diagram is often called Ferrers graph, in particular when it is drawn with dots instead of squares, see [1, Section 1.3].

For a nilpotent operator $N$ with nilpotency index $m$ the nonincreasing $m$-tuple $\nu = (\nu_1, \ldots, \nu_m)$ with $\nu_k = \dim((\ker N) \cap \text{ran}(N^{k-1}))$ for all $k \in \{1, \ldots, m\}$ is called the Weyr characteristic of $N$. Notice that the first entry equals the nullity of $N$ denoted by $d$: $\nu_1 = \dim \ker(N) = d$. The Weyr characteristic of $N$ determines the Young diagram with $m$ columns in which the $k$-th column has $\nu_k$ squares. The nonincreasing $d$-tuple $\mu = (\mu_1, \ldots, \mu_d)$ of numbers of squares in each row of this Young diagram is the Segre characteristic of $N$. We prove that a union of Jordan chains of $N$ forms a basis for $\mathfrak{F}$ if and only if the union consists of $d$ Jordan chains, the heads of the Jordan chains are linearly independent and the ordered lengths of the Jordan chains form the Segre characteristic of $N$.

To extend the characterization of a Jordan basis to an arbitrary operator on $\mathfrak{F}$, in Section 5 we give a direct proof of the Jordan decomposition theorem: An arbitrary operator on $\mathfrak{F}$ admits a Jordan decomposition. That is, for an operator $T$ on $\mathfrak{F}$ there exist a unique commuting diagonalizable operator $D$ and a nilpotent operator $N$ such that $T = D + N$. The Jordan decomposition theorem is commonly deduced from the existence of a Jordan basis, see [5] and [7, Section 8.15]. In this note we use Jordan decomposition in Section 6 to connect the considerations for a nilpotent operator to an arbitrary operator.

In Section 7 we present a canonical representation for an arbitrary operator on $\mathfrak{F}$ in, what we call, a canonical space of vector polynomials. In this canonical space the role of $D$ is played by a diagonal $d \times d$ matrix and the role of $N$ is played either by the differentiation operator or the operator of truncated multiplication by the independent variable.

2. A Crash Course on Young Diagrams

2.1. Four Functions. In this section we consider four functions $\text{Con}$, $\text{Der}$, $\text{Int}$ and $\text{Lev}$ from the set of nonincreasing tuples of positive integers to itself. Tuples of
positive integers commonly appear in the theory of integer partitions. The function $\text{Con}$ appears in this context as conjugation of partitions, see [1, Definition 1.8].

Let $d, m \in \mathbb{N}$ and let $\mu = (\mu_1, \ldots, \mu_d)$ be a $d$-tuple of positive integers such that $\mu_1 \geq \cdots \geq \mu_d$ with $\mu_1 = m$. For each $i \in \{1, \ldots, d\}$ let $a_{ij} \in \mathbb{C}$ for all $j \in \{1, \ldots, \mu_j\}$. How to change the order of summation in the sum

$$\sum_{i=1}^{d} \sum_{j=1}^{\mu_j} a_{ij} = \sum_{j=1}^{m} \sum_{i=1}^{\nu_j} a_{ij}.$$ 

To find out which positive integers should replace the question mark in the sum on the right hand side, we consider an example of a 5-tuple $\mu = (7, 6, 6, 3, 2)$ and illustrate it with a Young diagram in which the positive integers in $\mu$ are represented by rows of squares in the diagram below:

$$\begin{array}{ccccccc}
\text{a}_{11} & \text{a}_{12} & \text{a}_{13} & \text{a}_{14} & \text{a}_{15} & \text{a}_{16} & \text{a}_{17} \\
\text{a}_{21} & \text{a}_{22} & \text{a}_{23} & \text{a}_{24} & \text{a}_{25} & \text{a}_{26} \\
\text{a}_{31} & \text{a}_{32} & \text{a}_{33} & \text{a}_{34} & \text{a}_{35} & \text{a}_{36} \\
\text{a}_{41} & \text{a}_{42} & \text{a}_{43} \\
\text{a}_{51} & \text{a}_{52} \\
\end{array}$$

(2.1)

In this example, by counting the squares in each column in the Young diagram in (2.1) we see that with the 7-tuple $\nu = (5, 5, 4, 3, 3, 3, 1)$ we have

$$\sum_{i=1}^{5} \sum_{j=1}^{\mu_j} a_{ij} = \sum_{j=1}^{7} \sum_{i=1}^{\nu_j} a_{ij}.$$ 

The 7-tuple $\nu = (5, 5, 4, 3, 3, 3, 1)$ is said to be the conjugate tuple to the 5-tuple $\mu = (7, 6, 6, 3, 2)$.

In general, with $d, m \in \mathbb{N}$ for a nonincreasing $d$-tuple $\mu = (\mu_1, \ldots, \mu_d)$ of positive integers with $\mu_1 = m$ we define its conjugate tuple to be the $m$-tuple $\nu = (\nu_1, \ldots, \nu_m)$ with

$$\nu_k = \# \{ j \in \{1, \ldots, d\} : \mu_j \geq k \} = \max \{ j \in \{1, \ldots, d\} : \mu_j \geq k \} \text{ for all } k \in \{1, \ldots, m\},$$

(2.2)

where $\#$ denotes the cardinality of a finite set and the last equality is a consequence of the fact that $\mu$ is nonincreasing.

If the $d$-tuple $\mu$ is considered as a function $\mu : \{1, \ldots, d\} \to \{1, \ldots, m\}$, then its conjugate $m$-tuple $\nu : \{1, \ldots, m\} \to \{1, \ldots, d\}$ is often called a generalized inverse of $\mu$, see [4] and references therein.

Denote by $\mathcal{D}$ the set of all nonincreasing tuples of positive integers. Let $\text{Con} : \mathcal{D} \to \mathcal{D}$ be defined by: $\text{Con} \mu = \nu$ where $\nu$ is the conjugate tuple to $\mu \in \mathcal{D}$ defined in (2.2). Conjugation of a tuple represented by a Young diagram corresponds to transposition of the Young diagram. This geometric property implies that conjugation is an involution, see [1, Section 1.3]. For completeness, we provide a proof based on the formal definition in (2.2).
Proposition 2.1. The function $\text{Con}$ is an involution on $\mathcal{D}$, that is, $\text{Con}$ is a bijection and $\text{Con} = \text{Con}^{-1}$.

Proof. Let $d, m \in \mathbb{N}$, let $\mu = (\mu_1, \ldots, \mu_d) \in \mathcal{D}$ be a $d$-tuple with $\mu_1 = m$. Let the $m$-tuple $\nu = (\nu_1, \ldots, \nu_m)$ be the conjugate tuple to $\mu$. Then $\nu_1 = d$. Let $\sigma = \text{Con} \nu$, that is, $\sigma_j = \max\{k \in \{1, \ldots, m\} : \nu_k \geq j\}$ for all $j \in \{1, \ldots, d\}$. Consequently, $\sigma_j \leq m$ for all $j \in \{1, \ldots, m\}$.

By the definitions of $\nu$ and $\sigma$ for all $j, l \in \{1, \ldots, m\}$ the following four implications hold

$$
\mu_j \geq l \Rightarrow \nu_l \geq j \Rightarrow \sigma_j \geq l \quad \text{and} \quad \mu_j < l \Rightarrow \nu_l < j \Rightarrow \sigma_j < l.
$$

Let $j \in \{1, \ldots, m\}$ be arbitrary. If $\mu_j = m$, the two implications on the left and $\sigma_j \leq m$ yield $\sigma_j = m$. Assume $\mu_j < m$ and set $l = \mu_j$ in the two implications on the left to get $\sigma_j \geq \mu_j$. Setting $l = \mu_j + 1$ in the two implications on the right we get $\sigma_j < \mu_j + 1$, proving that $\sigma_j = \mu_j$. \hfill \Box

Let $\delta = (\delta_1, \ldots, \delta_m)$ be a decreasing $m$-tuple of positive integers. Set $\text{Der} \delta$ to be the $m$-tuple of differences, that is,

$$
\text{Der} \delta = \nu \quad \text{where} \quad \nu_k = \delta_k - \delta_{k+1} \quad \text{for all} \quad k \in \{1, \ldots, m\} \quad \text{with} \quad \delta_{m+1} = 0.
$$

A decreasing $m$-tuple $\delta$ of positive integers is said to be convex if $\text{Der} \delta$ is a non-increasing $m$-tuple of positive integers. By $\mathcal{C}$ we denote the set of all convex decreasing tuples of positive integers. With this notation, $\text{Der} : \mathcal{C} \to \mathcal{D}$ is a function with domain $\mathcal{C}$ and codomain $\mathcal{D}$.

The function $\text{Der} : \mathcal{C} \to \mathcal{D}$ is a bijection. This is proved by verifying that the inverse of $\text{Der}$ is $\text{Int} : \mathcal{D} \to \mathcal{C}$ defined by

$$
\text{Int} \nu = \delta \quad \text{where}, \quad \text{if} \quad \nu = (\nu_1, \ldots, \nu_m) \in \mathcal{D}, \quad \delta_k = \nu_k + \cdots + \nu_m \quad \text{for all} \quad k \in \{1, \ldots, m\}.
$$

For an arbitrary tuple $\mu = (\mu_1, \ldots, \mu_d)$ in $\mathcal{D}$ with $\mu_1 = m$ set $\text{Lev} \mu = \delta$ where $\delta = (\delta_1, \ldots, \delta_m)$ with

$$
\delta_k = \max_{j=1}^d \mu_j - (k-1), 0 \quad \text{for all} \quad k \in \{1, \ldots, m\}.
$$

Proposition 2.2. The function $\text{Lev} : \mathcal{D} \to \mathcal{C}$ is a bijection, $\text{Lev} = \text{Int} \circ \text{Con}$ and $\text{Lev}^{-1} = \text{Con} \circ \text{Der}$.

Proof. Let $\mu = (\mu_1, \ldots, \mu_d)$ be an arbitrary tuple in $\mathcal{D}$ such that $\mu_1 = m$. Let $\nu = \text{Con} \mu$ with $\nu = (\nu_1, \ldots, \nu_m) \in \mathcal{D}$. Since $\mu$ is a tuple of positive integers, for all $j \in \{1, \ldots, d\}$ and all $k \in \{1, \ldots, m\}$ we have

$$
\max\{\mu_j - (k-1), 0\} - \max\{\mu_j - k, 0\} = \begin{cases} 1 & \text{if} \quad \mu_j \geq k \\ 0 & \text{if} \quad \mu_j < k \end{cases}.
$$

By the preceding equality and the definition of $\nu$ in (2.2) we obtain

$$
\nu_k = \# \{j \in \{1, \ldots, d\} : \mu_j \geq k\} = \sum_{j=1}^d \left( \max\{\mu_j - (k-1), 0\} - \max\{\mu_j - k, 0\} \right)
$$

for all $k \in \{1, \ldots, m\}$. Consequently,

$$
\nu_k + \cdots + \nu_m = \sum_{j=1}^d \max\{\mu_j - (k-1), 0\} \quad \text{for all} \quad k \in \{1, \ldots, m\}.
$$
Hence, \( \text{Int} \nu = \text{Lev} \mu \), that is, \( \text{Lev} \mu = \text{Int}(\text{Con} \mu) \) for all \( \mu \in \mathcal{D} \). Thus \( \text{Lev} = \text{Int} \circ \text{Con} \) is a bijection as a composition of two bijections. Since \( \text{Con} \) is an idempotent and \( \text{Der} \) is the inverse of \( \text{Int} \), we have \( \text{Lev}^{-1} = \text{Con} \circ \text{Der} \). \( \square \)

**Remark 2.3.** In the previous proof we proved that \( \text{Int} \nu = \text{Lev} \mu \) provided that \( \mu \) and \( \nu \) are conjugate tuples in \( \mathcal{D} \). Since the first entry in the tuple \( \text{Int} \nu \) is \( \nu_1 + \cdots + \nu_m \) and the first entry in the tuple \( \text{Lev} \mu \) is \( \mu_1 + \cdots + \mu_d \), the equality \( \text{Int} \nu = \text{Lev} \mu \) yields that

\[
\nu_1 + \cdots + \nu_m = \mu_1 + \cdots + \mu_d
\]

whenever \( \mu \) and \( \nu \) are conjugate tuples in \( \mathcal{D} \).

We end this subsection with the commutative diagram showing the functions \( \text{Con}, \text{Der}, \text{Int} \) and \( \text{Lev} \).

![Diagram](image)

**Figure 1.** The commutative diagram with the bijections \( \text{Con}, \text{Der}, \text{Int} \) and \( \text{Lev} \)

### 2.2. The Tuple Sum and the Tuple Merge.** All tuples in the next two definitions are in \( \mathcal{D} \). Let \( d, d', d'' \in \mathbb{N} \).

A \( d \)-tuple \( \mu \) is said to be the **tuple merge** of a \( d' \)-tuple \( \mu' \) and a \( d'' \)-tuple \( \mu'' \) if \( d = d' + d'' \) and there exist a partition \( \{1, \ldots, d\} = \{1, \ldots, d'\} \cup \{d' + 1, \ldots, d' + d''\} \) and increasing bijections \( \delta' : \{1, \ldots, d'\} \to \mathbb{N} \) and \( \delta'' : \{d' + 1, \ldots, d' + d''\} \to \mathbb{N} \) such that \( \mu_{\delta'(j)} = \mu'_j \) for all \( j \in \{1, \ldots, d'\} \) and \( \mu_{\delta''(j)} = \mu''_j \) for all \( j \in \{d' + 1, \ldots, d' + d''\} \); we use the notation \( \mu = \mu' \circ \mu'' \). Informally, to get \( \mu \) we concatenate \( \mu' \) and \( \mu'' \) and order the result in the nonincreasing order.

A \( d \)-tuple \( \nu \) is said to be the **tuple sum** of a \( d' \)-tuple \( \nu' \) and a \( d'' \)-tuple \( \nu'' \) if \( d = \max\{d', d''\} \) and \( \nu_j = \nu'_j + \nu''_j \) for all \( j \in \{1, \ldots, d\} \) where \( \nu'_j = 0 \) or \( \nu''_j = 0 \) for all \( j > \min\{d', d''\} \); we use the notation \( \nu = \nu' + \nu'' \).

The algebraic structures \( (\mathcal{D}, \circ) \) and \( (\mathcal{D}, +) \) are commutative semigroups with binary operations \( \circ \) and \( + \); \( (\mathcal{C}, +) \) is a subsemigroup of \( (\mathcal{D}, +) \).

**Proposition 2.4.** The function \( \text{Con} : \mathcal{D} \to \mathcal{D} \) is an isomorphism between the semigroups \( (\mathcal{D}, \circ) \) and \( (\mathcal{D}, +) \). That is, for all \( \mu', \mu'' \in \mathcal{D} \) we have \( \text{Con}(\mu' \circ \mu'') = (\text{Con} \mu') + (\text{Con} \mu'') \).

**Proof.** In this paragraph we use the notation from the definition of the tuple merge. Let \( \mu' = (\mu'_1, \ldots, \mu'_d), \mu'' = (\mu''_1, \ldots, \mu''_{d''}) \) and \( \mu' \circ \mu'' = \mu = (\mu_1, \ldots, \mu_m) \). With \( \mu'_1 = m', \mu''_1 = m'', \mu_1 = m \), by the definition of the tuple merge we have \( m = \max\{m', m''\} \). Let \( \text{Lev} \mu' = \delta' = (\delta'_1, \ldots, \delta'_{m'}) \), \( \text{Lev} \mu'' = \delta'' = (\delta''_1, \ldots, \delta''_{m''}) \) and \( \text{Lev} \mu = \delta = (\delta_1, \ldots, \delta_m) \). Then by the definition of \( \text{Lev} \) for all \( k \in \{1, \ldots, m\} \) we
have
\[
\delta_k = \sum_{i=1}^{d} \max\{\mu_i - (k - 1), 0\}
\]
\[
= \sum_{i \in D'} \max\{\mu_i - (k - 1), 0\} + \sum_{i \in D''} \max\{\mu_i - (k - 1), 0\}
\]
\[
= \sum_{j=1}^{d'} \max\{\mu_j - (k - 1), 0\} + \sum_{j=1}^{d''} \max\{\mu_j - (k - 1), 0\}
\]
\[
= \delta_k^\prime + \delta_k^\prime\prime.
\]
Thus \(\text{Lev}(\mu' \circ \mu'\prime\prime) = \text{Lev}(\mu') + \text{Lev}(\mu'\prime\prime)\). Consequently, \(\text{Lev} : D \rightarrow C\) is an isomorphism between the semigroups \((D, \emptyset)\) and \((C, +)\).

Next we will prove that \(\text{Der} : C \rightarrow D\) is an isomorphism between the semigroups \((C, +)\) and \((D, +)\). We use the notation from the previous paragraph where \(\delta = \delta^\prime + \delta^\prime\prime\). Let \(\text{Der} \delta^\prime = \nu' = (\nu'_1, \ldots, \nu'_m), \text{Der} \delta^\prime\prime = \nu'' = (\nu''_1, \ldots, \nu''_m)\) and \(\text{Der} \delta = \nu = (\nu_1, \ldots, \nu_m)\). Then by the definition of \(\text{Der}\) for all \(k \in \{1, \ldots, m\}\) we have
\[
\nu_k = \delta_k - \delta_{k+1} = \delta_k^\prime - \delta_{k+1}^\prime + \delta_k^\prime\prime - \delta_{k+1}^\prime\prime = \nu'_k + \nu''_k.
\]
Since Proposition 2.2 implies that \(\text{Con} = \text{Der} \circ \text{Lev}\), the proposition is proved. \(\square\)

Merging ordered lists is a classical problem in computer programming. Proposition 2.4 offers a merge algorithm which reduces merge of two ordered tuples of positive integers to adding the conjugate tuples and then conjugating the result.

**Corollary 2.5.** For all \(\mu', \mu'' \in D\) we have \(\mu' \circ \mu'' = \text{Con}((\text{Con} \mu') + (\text{Con} \mu''))\).

**Remark 2.6.** The *Combinatorica* package in Wolfram *Mathematica* computer algebra system contains two relevant functions. One is the function *FerrersDiagram* which outputs the Young diagram with dots of a given tuple and the other one is *TransposePartition* which outputs the conjugate tuple of a given tuple, see [10, Section 4.1.3].

## 3. Linear Algebra Preliminaries

In this note \(n \in \mathbb{N}\) and \(\mathfrak{F}\) is an \(n\)-dimensional vector space over \(\mathbb{C}\). By \(\mathfrak{L}(\mathfrak{F})\) we denote the algebra of all linear operators defined on \(\mathfrak{F}\). The zero operator in \(\mathfrak{L}(\mathfrak{F})\) is denoted by \(0\) and the identity operator on \(\mathfrak{F}\) by \(I_\mathfrak{F}\); in both cases the subscript is omitted if the space is clear from the context. An operator \(N \in \mathfrak{L}(\mathfrak{F})\) is said to be \(\textit{nilpotent}\) if there exists an \(m \in \mathbb{N}\) such that \(N^m = 0\); the smallest \(m \in \mathbb{N}\) such that \(N^{m-1} \neq 0\) and \(N^m = 0\) is called the \(\textit{nilpotency index}\) of \(N\). By definition we set \(T^0 = I_\mathfrak{F}\) for all \(T \in \mathfrak{L}(\mathfrak{F})\).

Let \(T \in \mathfrak{L}(\mathfrak{F}), \lambda \in \mathbb{C}\) and \(v \in \mathfrak{F} \setminus \{0\}\). If \(Tv = \lambda v\), then \(\lambda\) is called an \(\textit{eigenvalue}\) of \(T\) and \(v\) is an \(\textit{eigenvector}\) corresponding to the eigenvalue \(\lambda\). The set of all eigenvalues of \(T\) is denoted by \(\sigma(T)\). The dimension of \(\ker(T - \lambda I)\) is called the \(\textit{geometric multiplicity}\) of \(\lambda \in \sigma(T)\). The sum \(\sum_{\lambda \in \sigma(T)} \dim \ker(T - \lambda I)\) is called the \(\textit{total geometric multiplicity}\) of \(\sigma(T)\). Let \(k \in \mathbb{N}\). A \(k\)-tuple of vectors \(u_1, \ldots, u_k \in \mathfrak{F} \setminus \{0\}\) is said to be a \(\textit{Jordan chain}\) of \(T\) of \(\textit{length}\) \(k\) corresponding
to $\lambda \in \sigma(T)$ if $Tu_j = \lambda u_j + u_{j+1}$ for all $j \in \{1, \ldots, k-1\}$ and $Tu_k = \lambda u_k$. The vector $u_k$ is said to be the head of the Jordan chain $u_1, \ldots, u_k$. A basis of $\mathfrak{F}$ which consists of Jordan chains of $T$ is called a Jordan basis of $\mathfrak{F}$ relative to $T$. If $T$ and $\mathfrak{F}$ are clear from the context, we simply call it a Jordan basis.

An operator $D \in \mathcal{L}(\mathfrak{F})$ is said to be diagonalizable if there exists a basis for $\mathfrak{F}$ which consists of eigenvectors of $D$, equivalently,

$$\mathfrak{F} = \bigoplus_{\lambda \in \sigma(D)} \ker(D - \lambda I), \quad \text{direct sum.} \quad (3.1)$$

In the following lemma we collect some facts from linear algebra.

**Lemma 3.1.** Let $n = \dim \mathfrak{F}$ and let $T \in \mathcal{L}(\mathfrak{F})$.

(a) There exists an $m \in \{1, \ldots, n\}$ such that for all $j \in \{1, \ldots, m\}$ we have $\ker(T^{j-1}) \subset \ker(T^j)$ and $\text{ran}(T^j) \subset \text{ran}(T^{j-1})$, proper subsets, and for all $k \in \mathbb{N}$ we have

$$\ker(T^m) = \ker(T^{m+k}) \quad \text{and} \quad \text{ran}(T^m) = \text{ran}(T^{m+k}).$$

(b) $\mathfrak{F} = \ker(T^n) \oplus \text{ran}(T^n)$, direct sum.

(c) $T$ is nilpotent if and only if $T^n = 0$.

For items (a) and (b) of Lemma 3.1 we refer to theorems and exercises in [2, Section 8A]; item (c) follows from (a).

**Proposition 3.2.** Let $D$ be a diagonalizable operator on $\mathfrak{F}$ and let $\mathfrak{G}$ be a subspace of $\mathfrak{F}$ which is invariant under $D$. Then the restriction $D|_{\mathfrak{G}}$ of $D$ to $\mathfrak{G}$ is diagonalizable on $\mathfrak{G}$.

**Proof.** Assume that $D$ is a diagonalizable operator on $\mathfrak{F}$ and $D\mathfrak{G} \subseteq \mathfrak{G}$. If $D$ has only one eigenvalue, then $D$ is a scalar multiple of $I$ and the proposition is clearly true. Otherwise, for each $\lambda \in \sigma(D)$ set

$$C_{\lambda} = \prod_{\zeta \in \sigma(D) \setminus \{\lambda\}} (D - \zeta I) \quad \text{and} \quad c_{\lambda} = \prod_{\zeta \in \sigma(D) \setminus \{\lambda\}} (\lambda - \zeta);$$

then $C_{\lambda} \in \mathcal{L}(\mathfrak{F})$ and $c_{\lambda} \in \mathbb{C} \setminus \{0\}$. Since $D\mathfrak{G} \subseteq \mathfrak{G}$, we have $C_{\lambda}\mathfrak{G} \subseteq \mathfrak{G}$ for every $\lambda \in \sigma(D)$. Clearly

$$C_{\lambda}v = c_{\lambda}v \quad \text{for all} \quad v \in \ker(D - \lambda I)$$

and

$$C_{\lambda}v = 0 \quad \text{for all} \quad v \in \ker(D - \zeta I) \quad \text{and} \quad \text{for all} \quad \zeta \in \sigma(D) \setminus \{\lambda\}.$$ 

Let $w \in \mathfrak{G}$. Since $D$ is diagonalizable, by (3.1), for every $\zeta \in \sigma(D)$ there exists a unique vector $v_{\zeta} \in \ker(D - \zeta I)$ such that $w = \sum_{\zeta \in \sigma(D)} v_{\zeta}$. Let $\lambda \in \sigma(D)$ be arbitrary. Applying $C_{\lambda}$ to the last equality we get $C_{\lambda}w = c_{\lambda}v_{\lambda}$. Since $C_{\lambda}w \in \mathfrak{G}$, we deduce that $v_{\lambda} \in \mathfrak{G}$ whenever $v_{\lambda} \neq 0$. Hence, for all $\lambda \in \sigma(D)$, $v_{\lambda} \neq 0$ implies $\lambda \in \sigma(D|_{\mathfrak{G}})$. Therefore, $w = \sum_{\lambda \in \sigma(D|_{\mathfrak{G}})} v_{\lambda}$. This proves that

$$\mathfrak{G} \subseteq \bigoplus_{\lambda \in \sigma(D|_{\mathfrak{G}})} \ker(D|_{\mathfrak{G}} - \lambda I_{\mathfrak{G}}), \quad \text{direct sum.}$$

Since the converse inclusion is trivial, the equality prevails. That is, by (3.1), $D|_{\mathfrak{G}}$ is diagonalizable. \qed
Lemma 3.3. Let $D$ be a diagonalizable operator on $\mathcal{F}$ and let $A$ be an operator on $\mathcal{F}$ which commutes with $D$. If $w \in (\text{ran } A) \cap \ker(D - \lambda I)$, then there exists $u \in \ker(D - \lambda I)$ such that $Au = w$.

Proof. The only relevant case is $\lambda \in \sigma(D)$. Assume $w \in (\text{ran } A) \cap \ker(D - \lambda I)$. Let $v \in \mathcal{F}$ be such that $w = Av$. Since $D$ is diagonalizable and by (3.1), for every $\zeta \in \sigma(D)$ there exists a unique vector $v_\zeta \in \ker(D - \zeta I)$ such that $v = \sum_{\zeta \in \sigma(D)} v_\zeta$. Since $A$ and $D$ commute, $Av_\zeta \in \ker(D - \zeta I)$ for every $\zeta \in \sigma(D)$. The linearity of $A$ yields $w = Av = \sum_{\zeta \in \sigma(D)} Av_\zeta$. Since $w \in \ker(D - \lambda I)$ and since the sum in (3.1) is direct, the equality in the preceding sentence implies that $Av_\zeta = 0$ for all $\zeta \in \sigma(D) \setminus \{\lambda\}$. That is, $w = Av_\lambda$ with $v_\lambda \in \ker(D - \lambda I)$. \ \qed

4. Two Linear Algebra Propositions involving Conjugate Tuples

The first two appearances of conjugate tuples in linear algebra in this paper are in the following proposition and in the proof of the next one. In the first proposition we use the Steinitz Exchange Lemma; for a proof see [13] and [6, page 122], where the lemma is attributed to H.G. Graßmann.

Lemma 4.1 (Steinitz Exchange Lemma). Let $\mathcal{A}$ and $\mathcal{B}$ be finite subsets of $\mathcal{F}$. If $\mathcal{A}$ is linearly independent and $\mathcal{B}$ spans $\mathcal{F}$, then $\#\mathcal{A} \leq \#\mathcal{B}$ and there exists a set $\mathcal{B}'$ such that $\mathcal{B}' \subseteq \mathcal{B}$, $\#\mathcal{B}' = \#\mathcal{B} - \#\mathcal{A}$ and $\mathcal{A} \cup \mathcal{B}'$ spans $\mathcal{F}$.

Proposition 4.2. Let $d, m \in \mathbb{N}$. Let $\mathcal{G}$ be a vector space with $d = \dim \mathcal{G}$ and let $D$ be a diagonalizable operator on $\mathcal{G}$. Let $\mathcal{G}_1, \ldots, \mathcal{G}_m$ be subspaces of $\mathcal{G}$ such that

\[
\mathcal{G} = \mathcal{G}_1 \supseteq \cdots \supseteq \mathcal{G}_m \supseteq \{0\} \tag{4.1}
\]

and $D\mathcal{G}_k \subseteq \mathcal{G}_k$ for all $k \in \{1, \ldots, m\}$. Set $\nu_k = \dim(\mathcal{G}_k)$. Then $\nu = (\nu_1, \ldots, \nu_m)$ is a nonincreasing $m$-tuple of positive integers with $\nu_1 = d$ and there exists a basis $\{w_1, \ldots, w_d\}$ of $\mathcal{G}$ which consists of eigenvectors of $D$ such that

\[
\mathcal{G}_k = \text{span}\{w_1, \ldots, w_{\nu_k}\} \quad \text{for all } \quad k \in \{1, \ldots, m\}. \tag{4.2}
\]

Furthermore, for any basis $\{w_1, \ldots, w_d\}$ satisfying (4.2) we have

\[
w_j \in \mathcal{G}_{\mu_j} \quad \text{and} \quad w_j \notin \mathcal{G}_{\mu_j+1} \quad \text{for all } \quad j \in \{1, \ldots, d\}, \tag{4.3}
\]

where $\mu = (\mu_1, \ldots, \mu_d)$ is the conjugate $d$-tuple to $\nu$ and $\mathcal{G}_{m+1} = \{0\}$.

Proof. If $m = 1$ the proposition is clear. Assume that $m \geq 2$, (4.1) holds, $\nu_k = \dim(\mathcal{G}_k)$ and $D\mathcal{G}_k \subseteq \mathcal{G}_k$ for all $k \in \{1, \ldots, m\}$. Then $d = \nu_1 \geq \cdots \geq \nu_m$. Since $D\mathcal{G}_m \subseteq \mathcal{G}_m$, Proposition 3.2 yields that there exists a basis $\mathcal{B}_m = \{w_1, \ldots, w_{\nu_m}\}$ for $\mathcal{G}_m$ which consists of eigenvectors of $D$. Let $k \in \{1, \ldots, m-1\}$. If $\nu_k = \nu_{k+1}$, then set $\mathcal{B}_k = \emptyset$. Now assume $\nu_k > \nu_{k+1}$. By Proposition 3.2 there exist a basis $\mathcal{A}_k$ for $\mathcal{G}_{k+1}$ which consists of $\nu_{k+1}$ eigenvectors of $D$ and a basis $\mathcal{A}_0$ for $\mathcal{G}_k$ which consists of $\nu_k$ eigenvectors of $D$. By Lemma 4.1 there exists a subset $\mathcal{A}_0'$ of $\mathcal{A}_0$ with $\nu_{k+1}$ elements such that $\mathcal{A}_1 \cup (\mathcal{A}_0 \setminus \mathcal{A}_0')$ is a basis for $\mathcal{G}_k$. Set

\[
\mathcal{B}_k = \mathcal{A}_0 \setminus \mathcal{A}_0' = \{w_{\nu_{k+1}+1}, \ldots, w_{\nu_k}\}.
\]

Then $\mathcal{B}_k$ consists of $\nu_k - \nu_{k+1}$ linearly independent eigenvectors of $D$ such that

\[
\mathcal{G}_k = \mathcal{G}_{k+1} \oplus \text{span}\mathcal{B}_k, \quad \text{direct sum}.
\]
In this way we have selected exactly
\[ \nu_m + \sum_{k=1}^{m-1} (\nu_k - \nu_{k+1}) = \nu_1 = d \]
linearly independent eigenvector \( w_1, \ldots, w_d \) of \( D \) such that
\[ \{w_1, \ldots, w_d\} = B_m \cup \cdots \cup B_1 \]
is a basis for \( \mathcal{G} \) which satisfies (4.2).

Let \( j \in \{1, \ldots, d\} \) be arbitrary. It follows from (4.2) that \( w_j \in \mathcal{G}_k \) if and only if \( \nu_k \geq j \). This equivalence and the definition of a conjugate tuple imply (4.3). □

**Proposition 4.3.** Let \( N \) be a nilpotent operator on \( \mathcal{G} \). Let \( d, m \in \mathbb{N} \), \( v_1, \ldots, v_d \in \mathcal{G} \) and let \( \mu_1, \ldots, \mu_d \in \mathbb{N} \) be such that
\[ m = \mu_1 \geq \cdots \geq \mu_d \quad \text{and} \quad N^{\mu_j-1}v_j \in \ker(N) \quad \text{for all} \quad j \in \{1, \ldots, d\}. \tag{4.4} \]
Then the vectors
\[ v_j, Nv_j, \ldots, N^{\mu_j-1}v_j, \quad j \in \{1, \ldots, d\}, \tag{4.5} \]
are linearly independent if and only if the vectors
\[ N^{\mu_1-1}v_1, \ldots, N^{\mu_d-1}v_d \tag{4.6} \]
are linearly independent.

**Proof.** Since the vectors listed in (4.6) are also listed in (4.5), we deduce that the linear independence of the vectors in (4.5) implies the linear independence of the vectors in (4.6). We prove the converse by using the conjugate \( m \)-tuple \( (\nu_1, \ldots, \nu_m) \) to \( (\mu_1, \ldots, \mu_d) \).

Assume (4.4) and assume that the set in (4.6) is linearly independent. To prove that the set in (4.5) is linearly independent let \( \alpha_{1,j}, \ldots, \alpha_{\mu_j,j} \in \mathbb{C} \) for all \( j \in \{1, \ldots, d\} \) and assume
\[ \sum_{i=1}^{d} \sum_{j=1}^{\mu_j} \alpha_{i,j} N^{\mu_j-i}v_j = 0. \]
Changing the order of summation in the preceding double sum, as explained in Section 2, we can rewrite the assumption as
\[ \sum_{j=1}^{d} \sum_{i=1}^{\mu_j} \alpha_{i,j} N^{\mu_j-i}v_j = 0. \tag{4.7} \]
Now we present a general recursive step which we will apply \( m \) times. Let \( l \in \{1, \ldots, m\} \). The following implication holds
\[ \sum_{i=1}^{l} \sum_{j=1}^{\nu_i} \alpha_{i,j} N^{\nu_j-i}v_j = 0 \quad \Rightarrow \quad \alpha_{l,j} = 0 \quad \text{for all} \quad j \in \{1, \ldots, \nu_l\}. \tag{4.8} \]
To prove (4.8), assume the hypothesis in (4.8) and apply the operator \( N^{l-1} \) to it. We get
\[ \sum_{i=1}^{l} \sum_{j=1}^{\nu_i} \alpha_{i,j} N^{l-1+\nu_j-i}v_j = 0. \]
Since we assume (4.4) and (4.6), we have \( N^{l-1+\mu_j-i}v_j \neq 0 \) if and only if \( l-1+\mu_j-i < \mu_j \), that is, if and only if \( l-1 < i \). As \( i \leq l \), we deduce that \( N^{l-1+\mu_j-i}v_j \neq 0 \)
if and only if $i = l$. Thus the hypothesis in (4.8) reads
$$
\sum_{j=1}^{\nu} \alpha_{l,j} N^{\nu_j - 1} v_j = 0.
$$
Since the vectors in (4.6) are linearly independent, the preceding equality implies the conclusion in (4.8).

The implication (4.8) with $l = m$ together with the assumption (4.7) yield $\alpha_{m,j} = 0$ for all $j \in \{1, \ldots, \nu_m\}$. Hence, the hypothesis of (4.8) with $l = m - 1$ is satisfied, and thus, $\alpha_{m-1,j} = 0$ for all $j \in \{1, \ldots, \nu_{m-1}\}$. Repeating this reasoning recursively for all $l \in \{m, \ldots, 1\}$ we get that $\alpha_{l,j} = 0$ for all $j \in \{1, \ldots, \nu_l\}$ and all $l \in \{1, \ldots, m\}$. This proves the linear independence of the vectors in (4.5). \hfill \Box

5. JORDAN DECOMPOSITION

Let $T \in \mathfrak{L}(\mathfrak{F})$. We call $T = D + N$ a Jordan decomposition of $T$ if $D, N \in \mathfrak{L}(\mathfrak{F})$ where $D$ is diagonalizable, $N$ is nilpotent and $ND = DN$.

In the next proposition we prove that the property of having a Jordan decomposition is stable under addition of a scalar operator and under taking powers of operators.

**Proposition 5.1.** Let $T \in \mathfrak{L}(\mathfrak{F})$, $\lambda \in \mathbb{C}$, and $l \in \mathbb{N}$.

(a) $T = D + N$ is a Jordan decomposition of $T$ with a diagonalizable $D$ and a nilpotent $N$ if and only if $T - \lambda I = (D - \lambda I) + N$ is a Jordan decomposition of $T - \lambda I$ with a diagonalizable $D - \lambda I$ and a nilpotent $N$.

(b) If $T = D + N$ is a Jordan decomposition of $T$ with a diagonalizable $D$ and a nilpotent $N$, then $T^l = D^l + N K_l$ is a Jordan decomposition of $T^l$ with a diagonalizable $D^l$ and a nilpotent $N K_l$ where

$$K_l = T^{l-1} + T^{l-2} D + \cdots + T D^{l-2} + D^{l-1}.$$

**Proof.** The equivalence in (a) follows from the fact that $D$ is diagonalizable if and only if $D - \lambda I$ is diagonalizable.

To prove (b) assume that $T = D + N$ is a Jordan decomposition of $T$ with a diagonalizable $D$ and a nilpotent $N$. Since $D$ and $D^l$ have the same eigenvectors, $D^l$ is diagonalizable. Since $T$ and $D$ commute, we have

$$T^l - D^l = (T - D)(T^{l-1} + T^{l-2} D + \cdots + T D^{l-2} + D^{l-1}) = N K_l$$

Since $T$, $N$, $D$ and $K_l$ commute, $N K_l$ is nilpotent and commutes with $D^l$. Therefore $T^l = D^l + N K_l$ is a Jordan decomposition of $T^l$. \hfill \Box

**Theorem 5.2.** Let $n = \dim \mathfrak{F}$ and $T \in \mathfrak{L}(\mathfrak{F})$. If $T = D + N$ is a Jordan decomposition of $T$ with a diagonalizable $D$ and a nilpotent $N$, then the following statements hold:

(a) For every $\lambda \in \mathbb{C}$ and for every $k \in \mathbb{N}$ we have

$$\ker((T - \lambda I)^k) = (\ker(N^k)) \cap \ker(D - \lambda I). \quad (5.1)$$

(b) For every $k \in \mathbb{N}$ we have

$$\ker(N^k) = \bigoplus_{\lambda \in \sigma(T)} \ker((T - \lambda I)^k), \quad \text{direct sum.} \quad (5.2)$$

(c) $D$ and $N$ are unique.

(d) An $l$-tuple of vectors $u_1, \ldots, u_l$ is a Jordan chain of $T$ corresponding to $\lambda \in \sigma(T)$ if and only if $u_j \in \ker(D - \lambda I)$, $u_j = N^{j-1} u_1$ for all $j \in \{1, \ldots, l\}$ and $N u_l = 0$, that is, $u_1, \ldots, u_l$ is a Jordan chain for $N$. 

Proof. (a) We start with proving the claim for $k = n$ and $\lambda = 0$. First we prove

$$\ker(T) \subseteq \ker(D) \subseteq \ker(T^n).$$

(5.3)

Let $Tx = 0$. Then $Nx = -Dx$. Consequently, $N^2x = -NDx = -DNx = D^2x$. Applying $N^2$ and $D^2$ to both sides of $N^2x = D^2x$ and using commutativity of $N$ and $D$ we get $N^4x = D^4x$. Repeating this process $N^jx = D^jx$ for all $j \in \mathbb{N}$. Since $N^jx = 0$ whenever $2^j \geq n$, we have that $D^jx = 0$ whenever $2^j \geq n$. Since $D$ is diagonalizable, we have $\ker(D^i) = \ker(D)$ for all $i \in \mathbb{N}$. Hence, $Dx = 0$.

Now we assume $Dx = 0$. Then, since by Lemma 3.1(c) $N^n x = 0$,

$$T^n x = (N + D)^n x = N^n x + \sum_{j=1}^{n} \binom{n}{j} N^{n-j} D^j x = 0.$$

Thus, $x \in \ker(T^n)$. This completes the proof of (5.3).

By Proposition 5.1(b) $T^n$ admits the Jordan decomposition $T^n = D^n + NK_n$. Applying (5.3) to this Jordan decomposition yields $\ker(T^n) \subseteq \ker(D^n) \subseteq \ker(T^{2n})$.

Since by Lemma 3.1(a) $\ker(T^{2n}) = \ker(T^n)$ and $\ker(D) = \ker(D^n)$ as $D$ is diagonalizable, we have $\ker(T^n) = \ker(D)$. Proposition 5.1(a) implies that for all $\lambda \in \mathbb{C}$ we have

$$\ker((T - \lambda I)^n) = \ker(D - \lambda I).$$

(5.4)

Since $N^n = 0$, (5.4) is (5.1) with $k = n$.

Let $k \in \mathbb{N}$ and $x \in \ker((T - \lambda I)^k)$ be arbitrary. Lemma 3.1(a) yields $x \in \ker((T - \lambda I)^n)$. By (5.4), $Dx = \lambda x$. Therefore, $Nx = (T - D)x = (T - \lambda I)x$.

Since $N$ and $T - \lambda I$ commute, we have $N^k x = (T - \lambda I)^k x = 0$. Hence, $x \in (\ker(N^k)) \cap \ker(D - \lambda I)$. This proves $\subseteq$ in (5.1). To prove the converse inclusion, assume $x \in (\ker(N^k)) \cap \ker(D - \lambda I)$. Then, as before, $Nx = (T - \lambda I)x$ and thus, $(T - \lambda I)^k x = N^k x = 0$. This completes the proof of (a).

(b) Let $k \in \mathbb{N}$ be arbitrary. Since $N$ and $D$ commute, $\ker(N^k)$ is invariant under $D$, by Proposition 3.2 the restriction $D|_{\ker(N^k)}$ is diagonalizable and, using (a),

$$\ker(D|_{\ker(N^k)} - \lambda I) = (\ker(N^k)) \cap \ker(D - \lambda I) = \ker((T - \lambda I)^k).$$

Consequently, $\sigma(D|_{\ker(N^k)}) = \sigma(T)$. Now (5.2) follows from (3.1).

(c) With $k = n$ (5.2) reads

$$\mathfrak{F} = \bigoplus_{\lambda \in \sigma(T)} \ker((T - \lambda I)^n), \quad \text{direct sum.}$$

(5.5)

Since $D$ and $(T - \lambda I)^n$ commute, (5.5) and (5.4) imply that $D$ is uniquely determined by the formula

$$D = \bigoplus_{\lambda \in \sigma(T)} \lambda I_{\ker((T - \lambda I)^n)}, \quad \text{direct sum of operators.}$$

As $N = T - D$, the uniqueness of $D$ implies uniqueness of $N$.

(d) Assume that an $l$-tuple $u_1, \ldots, u_l \in \mathfrak{F}$ is a Jordan chain of $T$ corresponding to $\lambda \in \sigma(T)$. Then $u_i = (T - \lambda)^{i-1} u_1$ for all $j \in \{1, \ldots, l\}$ and $(T - \lambda I)u_l = 0$. Therefore $(T - \lambda I)^j u_j = 0$ for all $j \in \{1, \ldots, l\}$. By (a) $u_j \in \ker(D - \lambda I)$ for all $j \in \{1, \ldots, l\}$. The proofs of the remaining statements in this item are straightforward. \qed
Remark 5.3. Consider $T \in \mathcal{L}(\mathfrak{F})$ with the Jordan decomposition $T = N + D$. It follows from Theorem 5.2(b) that the positive integers $\dim \text{ran}(N^k)$ with $k \in \mathbb{N}$, which play an important role in Section 6, can be expressed in terms of the operator $T$ as follows
\[
\dim \text{ran}(N^{k-1}) = n - \dim \ker(N^{k-1}) = n - \sum_{\lambda \in \sigma(T)} \dim \ker((T - \lambda I)^{k-1}) = n - \sum_{\lambda \in \sigma(T)} (n - \dim \text{ran}((T - \lambda I)^{k-1})) = n(1 - \#\sigma(T)) + \sum_{\lambda \in \sigma(T)} \dim \text{ran}((T - \lambda I)^{k-1}).
\]

Theorem 5.4. Every $T \in \mathcal{L}(\mathfrak{F})$ admits a Jordan decomposition.

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $T$. Then 0 is an eigenvalue of $S = T - \lambda I$. We will proceed by proving the theorem for $S$, relying on Proposition 5.1(a) to yield the general theorem for $T$.

The proof proceeds by Mathematical Induction on the dimension $n$ of the finite-dimensional space $\mathfrak{F}$. Since the only operator in a one-dimensional vector space which has zero as an eigenvalue is the zero operator, the theorem is valid for $n = 1$. Let $n \in \mathbb{N} \setminus \{1\}$ be arbitrary and assume that the theorem is valid for all vector spaces with dimension less than $n$.

Assume that $\dim \mathfrak{F} = n$ and set $\mathfrak{R} = \text{ran}(S)$. By the Nullity-Rank Theorem $n = \dim(\ker(S)) + \dim \mathfrak{R}$. Since $\ker(S) \neq \{0\}$, we have $m = \dim \mathfrak{R} < n$. Denote by $S_r \in \mathcal{L}(\mathfrak{R})$ the restriction of $S$ to $\mathfrak{R}$. By the inductive hypothesis $S_r$ has a Jordan decomposition $S_r = D_r + N_r$ with a diagonalizable $D_r \in \mathcal{L}(\mathfrak{R})$ and a nilpotent $N_r \in \mathcal{L}(\mathfrak{R})$. By Theorem 5.2(a) we have $\ker(S_r^{m-1}) = \ker(D_r)$; here we apply (5.1) to the Jordan decomposition $S_r = N_r + D_r$ in the vector space $\mathfrak{R}$ with $\lambda = 0$ and $k = n - 1 \geq m = \dim \mathfrak{R}$. Since clearly $S(\ker(S^n)) = \ker(S^n)$, we have $S(\ker(S^n)) = \ker(D_r)$ and consequently $\ker(S^n) = S^{-1}(\ker(D_r))$. As $\mathfrak{F}$ is by Lemma 3.1(b) a direct sum of $\ker(S^n)$ and $\text{ran}(S^n) \subseteq \text{ran} S = \mathfrak{R}$, we have
\[
\mathfrak{F} = \mathfrak{R} + S^{-1}(\ker(D_r)).
\]

Consequently, there exists a subspace $\mathfrak{C} \subset S^{-1}(\ker(D_r))$ such that
\[
\mathfrak{F} = \mathfrak{R} \oplus \mathfrak{C}, \quad \text{direct sum.}
\] (5.6)

We use the direct sum in (5.6) to define $D$ and $N$ for the Jordan decomposition of $S$ as follows. Let $v \in \mathfrak{F}$ be arbitrary. By (5.6) there exist a unique $x \in \mathfrak{R}$ and a unique $y \in \mathfrak{C}$ such that $v = x + y$. We set
\[
Dv = D_r x \quad \text{and} \quad Nv = N_r x + Sy.
\]

Since $S_r = D_r + N_r$, we have $S = D + N$ and $D$ is diagonalizable as $D_r$ is diagonalizable. To prove that $ND = DN$ we use the notation as in the definition of $D$ and $N$ and calculate
\[
NDv = N(D_r x + N_r x) = D_r N_r x + D(N_r x + Sy) = DNv,
\]
where, in the equality preceding the last one, we used the fact that $\mathfrak{C} \subseteq \ker(D_r) \subseteq \ker(D)$. To prove that $N^n = 0$ we recall that $Sy \in \mathfrak{R}$ and $N_r^{n-1} = 0$ and calculate
\[
N^n v = N^{n-1}(N_r x + Sy) = N_r^{n-1}(N_r x + Sy) = 0. \quad \square
6. JORDAN BASES

In the following theorem we prove that for an arbitrary $T \in \mathcal{L}(\mathfrak{g})$ there exists a Jordan basis for $\mathfrak{g}$ relative to $T$.

**Theorem 6.1 (Existence).** Let $n = \dim \mathfrak{g}$ and $T \in \mathcal{L}(\mathfrak{g})$. Let $T = D + N$ be the Jordan decomposition of $T$ with a diagonalizable $D$ and a nilpotent $N$. Let $d, m \in \{1, \ldots, n\}$ be such that

$$d = \dim \ker(N), \quad N^{m-1} \neq 0 \quad \text{and} \quad N^m = 0.$$  

For $k \in \{1, \ldots, m\}$ set

$$\omega_k = \rho_k - \rho_{k+1} \quad \text{where} \quad \rho_k = \dim \text{ran}(N^{k-1}) \quad \text{and} \quad \rho_{m+1} = 0.$$  

Then $\omega_1 \geq \cdots \geq \omega_m$ with $\omega_1 = d$. Let $\sigma = (\sigma_1, \ldots, \sigma_d)$ be the conjugate $d$-tuple to $\omega$. There exist vectors $u_1, \ldots, u_d \in \mathfrak{g}$ which are eigenvectors of $D$ and such that the vectors

$$N^{\sigma_1-1}u_1, \ldots, N^{\sigma_d-1}u_d$$

form a basis for $\ker(N)$ and the union of the following $d$ Jordan chains of $T$

$$u_j, Nu_j, \ldots, N^{\sigma_j-1}u_j, \quad j \in \{1, \ldots, d\},$$

form a basis for $\mathfrak{g}$.

**Proof.** By Lemma 3.1(a) and the definition of $m$ we have

$$\mathfrak{g} = \text{ran}(N^0) \supset \text{ran}(N) \supset \cdots \supset \text{ran}(N^{m-1}) \supset \{0\}. \quad (6.1)$$

Next we prove that $\omega$ is a nonincreasing $m$-tuple of positive integers. Let $k \in \{1, \ldots, m\}$ and consider the restriction $N|_{\text{ran}(N^{k-1})}$ of $N$ to $\text{ran}(N^{k-1})$. Then

$$\text{ran}(N|_{\text{ran}(N^{k-1})}) = \text{ran}(N^k)$$

and consequently the Nullity-Rank Theorem applied to the space $\text{ran}(N^{k-1})$ and the operator $N|_{\text{ran}(N^{k-1})}$ yields

$$\omega_k = \dim \text{ran}(N^{k-1}) - \dim \text{ran}(N^k) = \dim \ker(N|_{\text{ran}(N^{k-1})}).$$

In particular, $\omega_1 = d$.

Set $\mathfrak{g}_k = \ker(N|_{\text{ran}(N^{k-1})})$. Then $\mathfrak{g}_k = (\ker(N) \cap \text{ran}(N^{k-1})$ and (6.1) yields

$$\ker(N) = \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots \supset \mathfrak{g}_m = \text{ran}(N^{m-1}) \supset \{0\}. \quad (6.2)$$

Since for all $k \in \{1, \ldots, m\}$ we have $\omega_k = \dim(\mathfrak{g}_k)$, (6.2) yields that $\omega$ is a nonincreasing $m$-tuple of positive integers.

In the notation of Section 2, we just proved that $\rho \in \mathcal{C}$, $\omega \in \mathcal{D}$ and $\omega = \text{Der} \, \rho$, or, equivalently $\rho = \text{Int} \, \omega$. In the statement of the theorem we also introduced the $d$-tuple $\sigma = \text{Con} \, \omega = \text{Lev}^{-1} \rho$ which we will encounter in the next paragraph.

Proposition 4.2 applied to the vector space $\mathfrak{g} = \ker(N)$, the restriction of $D$ to $\ker(N)$ and the nonincreasing $m$-tuple of subspaces $\mathfrak{g}_k$ implies that there exists a basis $\{w_1, \ldots, w_d\}$ of $\ker(N)$ which consists of eigenvectors of $D$ such that (4.2) holds and for every $j \in \{1, \ldots, d\}$ we have

$$w_j \in \mathfrak{g}_{\sigma_j} = (\ker(N) \cap \text{ran}(N^{\sigma_j-1})).$$

Now Lemma 3.3 implies that there exist $u_1, \ldots, u_d \in \mathfrak{g}$ which are eigenvectors of $D$ and such that

$$w_j = N^{\sigma_j-1}u_j \quad \text{for all} \quad j \in \{1, \ldots, d\}.$$
Since \( \{w_1, \ldots, w_d\} \) is a basis for \( \ker(N) \), Proposition 4.3 implies that the vectors
\[
u_j, \ldots, N^{\sigma_j-1}u_j, \quad j \in \{1, \ldots, d\},
\]
are linearly independent. There are \( \sigma_1 + \cdots + \sigma_d \) vectors listed in (6.3). Remark 2.3 and the fact that \( \rho = \mathrm{Int} \omega = \mathrm{Lev} \sigma \) yield
\[
\sigma_1 + \cdots + \sigma_d = \omega_1 + \cdots + \omega_m = \rho_1 = n.
\]
As \( n = \dim \mathfrak{g} \), the linearly independent vectors in (6.3) form a basis for \( \mathfrak{g} \).

Since \( D \) and \( N \) commute and for every \( j \in \{1, \ldots, d\} \) \( u_j \) is an eigenvector of \( D \), there exists an eigenvalue \( \lambda_j \) of \( D \) such that \( u_j, \ldots, N^{\sigma_j-1}u_j \in \ker(D - \lambda_j I) \). Therefore for every \( j \in \{1, \ldots, d\} \) the vectors \( u_j, \ldots, N^{\sigma_j-1}u_j \) form a Jordan chain of \( T \). The theorem is proved. \( \Box \)

In Theorem 6.1 the tuples
\[
\rho, \quad \omega = \mathrm{Der} \rho \quad \text{and} \quad \sigma = \mathrm{Lev}^{-1} \rho
\]
will be called the range characteristic, the Weyr characteristic and the Segre characteristic of \( N \) and also of \( T \). When we want to emphasize the dependence of a characteristic on \( T \) we will write \( \rho_T, \omega_T \) and \( \sigma_T \) for its characteristics.

**Remark 6.2.** Commonly the Weyr and Segre characteristics are defined for each eigenvalue of an operator, see [8, Sections 29, 40], [11, Section 4.5], [12, Section VI.9], [14, Section 4.2]. So, our definitions of the Weyr and Segre characteristics for a nilpotent operator are equivalent to the definitions in [11, Section 4.2]. For a general operator with multiple distinct eigenvalues our definition of the Weyr characteristic is the tuple sum of the Weyr characteristics for individual eigenvalues, while our definition of the Segre characteristic is the tuple merge of the Segre characteristics for individual eigenvalues, see Subsection 2.2. A similar definition of the Segre characteristic of an arbitrary square matrix is given in [8, Section 29]. Further references and historical remarks can be found in [9].

**Remark 6.3.** Remark 5.3 yields that the range characteristics (and hence the other two characteristics as well) of \( T \) can be expressed directly in terms of \( T \). Also, (5.2) leads to the formulas for \( d \) and \( m \) in terms of \( T \); with \( k = 1 \) in (5.2) we have
\[
d = \sum_{\lambda \in \sigma(T)} \dim \ker(T - \lambda I),
\]
whence \( d \) is the total geometric multiplicity of \( \sigma(T) \), and (5.2) implies
\[
m = \min \left\{ k \in \mathbb{N} : n = \sum_{\lambda \in \sigma(T)} \dim \ker((T - \lambda I)^k) \right\}.
\]

The following corollary is a consequence of considerations in Section 2 and Theorem 6.1.

**Corollary 6.4.** Let \( S, T \in \mathcal{L}(\mathfrak{g}) \) and let \( S = D_S + N_S \) and \( T = D_T + N_T \) be the Jordan decompositions of \( S \) and \( T \), respectively. The following statements are equivalent: (a) \( \rho_S = \rho_T \); (b) \( \omega_S = \omega_T \); (c) \( \sigma_S = \sigma_T \); (d) \( N_S \) and \( N_T \) are similar.

**Theorem 6.5** (Uniqueness). Let \( T \in \mathcal{L}(\mathfrak{g}) \) and let \( d \) be the total geometric multiplicity of its spectrum. A union of Jordan chains of \( T \) forms a basis for \( \mathfrak{g} \) if and only if the union consists of \( d \) Jordan chains, the heads of the Jordan chains are...
linearly independent and the lengths of the Jordan chains in nonincreasing order form the Segre characteristic of $T$.

**Proof.** To prove the “if” part, let $T = D + N$ be the Jordan decomposition of $T$ with a diagonalizable $D$ and a nilpotent $N$. Let $\sigma = (\sigma_1, \ldots, \sigma_d)$ be the Segre characteristic of $T$. To represent Jordan chains of $T$ we use Theorem 5.2(d); let

$$u_j, Nu_j, \ldots, N^{\sigma_j-1}u_j, \quad j \in \{1, \ldots, d\},$$

be $d$ Jordan chains of $T$ whose heads $N^{\sigma_1-1}u_1, \ldots, N^{\sigma_d-1}u_d$ are linearly independent. By Proposition 4.3 the vectors in (6.4) are linearly independent. By Remark 2.3 and the definition of the Segre characteristic the number of vectors in (6.4) is $\sigma_1 + \cdots + \sigma_d = \rho_1 = \dim \mathfrak{F}$. Hence the vectors in (6.4) form a Jordan basis for $\mathfrak{F}$.

To prove the converse, let $p \in \mathbb{N}$, $v_1, \ldots, v_p \in \mathfrak{F}$ and let $\xi = (\xi_1, \ldots, \xi_p)$ be a $p$-tuple in $\mathfrak{F}$. Assume that $p$ Jordan chains of $T$

$$v_j, Nv_j, \ldots, N^{\xi_j-1}v_j, \quad j \in \{1, \ldots, p\},$$

form a Jordan basis for $\mathfrak{F}$. If $\xi = (1, \ldots, 1)$, then $\{v_1, \ldots, v_p\}$ is a basis for $\mathfrak{F}$ which consists of eigenvectors of $T$. Thus $T$ is diagonalizable and $p = \dim \mathfrak{F} = d$. The Jordan decomposition of $T$ is $T = T + 0$. Since by definition $(0\mathfrak{F})^0 = I_{\mathfrak{F}}$, the range and Weyr characteristics of 0 is the 1-tuple $(n)$, where $n = \dim \mathfrak{F}$. Consequently, in this case, $\xi$ is the Segre characteristic of $T$.

Assume that $\xi_1 > 1$. Let $u \in \ker(N)$ be arbitrary. Since the vectors in (6.5) form a basis for $\mathfrak{F}$, there exist $\alpha_{ij} \in \mathbb{C}$ such that $u = \sum_{i=1}^{p} \sum_{j=1}^{\xi_i} \alpha_{ij} N^{j-1}v_i$. Applying $N$ to the preceding equality and using that $N^{\xi_i-1}v_i \in \ker(N)$ we obtain

$$0 = \sum_{i=1}^{q} \sum_{j=1}^{\xi_i-1} \alpha_{ij} N^{j}v_i,$$

where $q = \max\{i \in \{1, \ldots, p\} : \xi_i > 1\}$. Since the vectors in (6.5) are linearly independent, the preceding displayed equality yields that for each $i \in \{1, \ldots, p\}$ such that $\xi_i > 1$ we have $\alpha_{ij} = 0$ for all $j \in \{1, \ldots, \xi_i - 1\}$. Therefore, $u = \sum_{i=1}^{p} \alpha_{i1} N^{\xi_i-1}v_i$. Since $u \in \ker(N)$ was arbitrary, and since the vectors in (6.5) are linearly independent we deduce that $\{N^{\xi_1-1}v_1, \ldots, N^{\xi_p-1}v_p\}$ is a basis for $\ker(N)$. Consequently, $p = \dim \ker(N) = d$.

Let $m$ be the index of nilpotency of $N$. Since $\xi \in \mathfrak{F}$, we have $\xi_1 \geq \xi_k$ for all $k \in \{1, \ldots, d\}$. Therefore, since the vectors in (6.5) form a basis for $\mathfrak{F}$, we have $N^{\xi_1-1} \neq 0$ and $N^{\xi_i} = 0$. Hence $\xi_1 = m$.

To prove that $\xi$ is the Segre characteristic of $N$ we calculate the dimensions $\rho_k = \dim \text{ran}(N^{k-1})$ for all $k \in \{1, \ldots, m\}$. That is, we will calculate the range characteristic $\rho = (\rho_1, \ldots, \rho_m)$ of $N$ and $T$.

Let $k \in \{1, \ldots, m\}$ and $v \in \text{ran}(N^{k-1})$ be arbitrary. Since the vectors in (6.5) form a basis for $\mathfrak{F}$, there exist $\alpha_{ij} \in \mathbb{C}$ such that $v = \sum_{i=1}^{d} \sum_{j=1}^{\xi_i} \alpha_{ij} N^{j-1}v_i$. Applying $N^{m-(k-1)}$ to the preceding equality and using $N^{\xi_i-1}v_j \in \ker(N)$ we get

$$0 = \sum_{i=1}^{d} \sum_{j=1}^{\xi_i} \alpha_{ij} N^{m+j-k}v_i = \sum_{i=1}^{d} \sum_{j=1}^{\xi_i+k-1-m} \alpha_{ij} N^{m+j-k}v_i$$

Since the vectors in (6.5) are linearly independent, the preceding equality yields that for each $i \in \{1, \ldots, d\}$ such that $\xi_i > m - (k - 1)$ we have $\alpha_{ij} = 0$ for all
\( j \in \{1, \ldots, \xi_i + k - 1 - m\} \). Therefore,

\[
v = \sum_{i=1}^{d} \sum_{j=k+\xi_i-m}^{\xi_i} \alpha_{ij} N^{j-1} v_i. \tag{6.6}
\]

As \( v \) was an arbitrary vector in \( \text{ran}(N^{k-1}) \), (6.6) implies that the set of vectors which consists of the last

\[
\max\{\xi_i - (k - 1), 0\} \ 	ext{vectors from} \ Nv_1, \ldots, N^{\xi_i-1}v_i \ 	ext{for all} \ i \in \{1, \ldots, d\}
\]

form a basis for \( \text{ran}(N^{k-1}) \). Therefore, the following formulas hold

\[
\rho_k = \dim \text{ran}(N^{k-1}) = \sum_{i=1}^{d} \max\{\xi_i - (k - 1), 0\} \ 	ext{for all} \ k \in \{1, \ldots, m\}.
\]

By the definition of the function \( \text{Lev} \), the preceding formulas show \( \rho = \text{Lev} \xi \). Since by Proposition 2.2 \( \text{Lev} \) is a bijection, we have \( \xi = \text{Lev}^{-1} \rho \). Thus the tuple \( \xi \) is the Segre characteristic of \( N \) and \( T \).

\section{Canonical Representations}

Let \( d \in \mathbb{N} \). By \( \mathbb{C}^d[z] \) we denote the vector space of all vector polynomials whose coefficients are the vectors in \( \mathbb{C}^d \). The vectors in the standard basis of \( \mathbb{C}^d \), that is, the columns of the identity matrix \( I_d \) we denote by \( e_{d,1}, \ldots, e_{d,d} \).

Let \( \mu = (\mu_1, \ldots, \mu_d) \) be a nonincreasing \( d \)-tuple of positive integers with \( \mu_1 = m \).

In this section we study a special subspace \( \mathcal{C}_\mu \) of the space of vector polynomials \( \mathbb{C}^d[z] \) associated with \( \mu \) defined as follows

\[
\mathcal{C}_\mu = \left\{ \begin{bmatrix} p_1(z) \\ \vdots \\ p_d(z) \end{bmatrix} \in \mathbb{C}^d[z] : p_k(z) \in \mathbb{C}[z], \ \deg p_k < \mu_k, \ k \in \{1, \ldots, d\} \right\}.
\]

The space \( \mathcal{C}_\mu \) is called a canonical subspace of \( \mathbb{C}^d[z] \), or simply a canonical space of vector polynomials.

The set of vector polynomials

\[
\{ z^{\ell} e_{d,k} : \ell \in \{0, 1, \ldots, \mu_k - 1\}, \ k \in \{1, \ldots, d\} \}
\]

is called the standard basis for \( \mathcal{C}_\mu \). Clearly

\[
\dim \mathcal{C}_\mu = \mu_1 + \cdots + \mu_d.
\]

Let \( \nu = (\nu_1, \ldots, \nu_m) \) be the conjugate tuple to \( \mu \). The polynomials in \( \mathcal{C}_\mu \) can be characterized using \( \nu \) in the following way. Let \( f(z) \in \mathbb{C}^d[z] \) be written as a polynomial with vector coefficients

\[
f(z) = f_0 + f_1 z + \cdots + f_{m-1} z^{m-1} \ 	ext{with} \ f_{k-1} \in \mathbb{C}^d \ 	ext{for all} \ k \in \{1, \ldots, m\}.
\]

Then \( f(z) \in \mathcal{C}_\mu \) if and only if \( f_{k-1} \in \text{span}\{e_{d,1}, \ldots, e_{d,\nu_k}\} \) for all \( k \in \{1, \ldots, m\} \).

In particular, if \( f(z) \in \mathcal{C}_\mu \), then the vector \( f_0 \) can be any vector in \( \mathbb{C}^d \) since \( \nu_1 = d \).

In other words, we have

\[
\mathcal{C}_\mu = \left\{ \sum_{k=1}^{m} f_{k-1} z^{k-1} : f_{k-1} \in \text{span}\{e_{d,1}, \ldots, e_{d,\nu_k}\} \ 	ext{for all} \ k \in \{1, \ldots, m\} \right\}.
\]
A prominent nilpotent operator on the canonical space $\mathcal{E}_\mu$ is the differentiation operator which we denote by $\partial_{\mathcal{E}_\mu}$. For $l \in \{1, \ldots, m\}$ we have

$$\dim \text{ran}(\partial_{\mathcal{E}_\mu}^{l-1}) = \dim \text{ran}(\partial_{\mathcal{E}_\mu}^l) = \dim \text{ran}(\partial_{\mathcal{E}_\mu}) = \dim \text{ran}(\partial_{\mathcal{E}_\mu})$$

Consequently,

$$\dim \text{ran}(\partial_{\mathcal{E}_\mu}^{l-1}) = \dim \text{ran}(\partial_{\mathcal{E}_\mu}^l) = \dim \text{ran}(\partial_{\mathcal{E}_\mu}) = \dim \text{ran}(\partial_{\mathcal{E}_\mu})$$

The preceding equalities, Proposition 2.2 and the definition of the Segre characteristic yield that $\mu$ is the Segre characteristic of the differentiation operator $\partial_{\mathcal{E}_\mu}$.

Another prominent nilpotent operator on $\mathcal{E}_\mu$ is obtained as a natural extension of the operator of multiplication by the independent variable $z$. Since multiplication by $z$ is not defined for all $p(z) \in \mathcal{E}_\mu$, we will introduce the operation of degree truncation of a scalar polynomial. Let $k \in \mathbb{N}$. Notice that for each scalar polynomial $a(z) \in \mathbb{C}[z]$ there exist unique polynomials $b(z), c(z) \in \mathbb{C}[z]$ such that $\deg b < k$ and

$$a(z) = b(z) + z^k c(z).$$

The polynomial $b(z)$ is the truncation of the polynomial $a(z)$ to a degree smaller than $k$; we denote it by $[a(z)]_{< k}$.

We define the nilpotent extension $Z_{\mathcal{E}_\mu} : \mathcal{E}_\mu \to \mathcal{E}_\mu$ of the operator of multiplication by $z$ on $\mathcal{E}_\mu$ by

$$(Z_{\mathcal{E}_\mu} f)(z) = \left[[zf_1(z)]_{< \mu_1} \cdots [zf_d(z)]_{< \mu_d}\right]^\top$$

for an arbitrary $f(z) = [f_1(z) \cdots f_d(z)]^\top \in \mathcal{E}_\mu$. Straightforward calculations show that for all $l \in \{1, \ldots, m\}$ we have

$$\dim \text{ran}(Z_{\mathcal{E}_\mu}^{l-1}) = \dim \text{ran}(Z_{\mathcal{E}_\mu}^l) = \dim \text{ran}(Z_{\mathcal{E}_\mu}) = \dim \text{ran}(Z_{\mathcal{E}_\mu})$$

Consequently,

$$\dim \text{ran}(Z_{\mathcal{E}_\mu}^{l-1}) = \dim \text{ran}(Z_{\mathcal{E}_\mu}^l) = \dim \text{ran}(Z_{\mathcal{E}_\mu}) = \dim \text{ran}(Z_{\mathcal{E}_\mu})$$

The preceding equalities and the definition of the Weyr characteristic imply that $\nu$ is the Weyr characteristic of $Z_{\mathcal{E}_\mu}$. Hence, $\mu$ is the Segre characteristic of the operator $Z_{\mathcal{E}_\mu}$.

**Theorem 7.1.** Let $\mathfrak{F}$ be a finite-dimensional vector space and let $T$ be an operator defined on all of $\mathfrak{F}$. Let $T = D + N$ be the Jordan decomposition of $T$ where $D$ is diagonalizable, $N$ is nilpotent and both commute with $T$. Let $d = \dim \text{ker}(N)$ and let $\mu = (\mu_1, \ldots, \mu_d)$ be the Weyr characteristic of $N$. Let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of $T$ each repeated according to its geometric multiplicity. There exist linear isomorphisms $\Phi : \mathfrak{F} \to \mathcal{E}_\mu$ and $\Psi : \mathfrak{F} \to \mathcal{E}_\mu$ such that

$$\Phi T \Phi^{-1} = \text{diag}(\lambda_1, \ldots, \lambda_d) + \partial_{\mathcal{E}_\mu} \quad \text{and} \quad \Psi T \Psi^{-1} = \text{diag}(\lambda_1, \ldots, \lambda_d) + Z_{\mathcal{E}_\mu}.$$
Proof. Let \( \{u_l, \ldots, N^{\mu_l-1}u_l : l \in \{1, \ldots, d\}\} \) be the basis for \( \mathfrak{H} \) whose existence was proved in Theorem 6.1. Let \( \lambda_1, \ldots, \lambda_d \) be the eigenvalues of \( T \) such that for all \( l \in \{1, \ldots, d\} \) we have \( u_l, \ldots, N^{\mu_l-1}u_l \in \ker(D - \lambda_l I) \).

Then
\[
TN^{k-1}w_l = \lambda_l N^{k-1}w_l + N^Kw_l \quad \text{for all} \quad k \in \{1, \ldots, \mu_l\} \quad \text{and} \quad l \in \{1, \ldots, d\}. \tag{7.1}
\]

The function defined by
\[
\Phi N^{k-1}u_l = \frac{1}{(\mu_l-k)!}e_{d,l}z^{\mu_l-k} \quad \text{for all} \quad k \in \{1, \ldots, \mu_l\} \quad \text{and} \quad l \in \{1, \ldots, d\}
\]
extends linearly to a linear bijection \( \Phi : \mathfrak{H} \to \mathfrak{C}^\mu_l \). Using this definition of \( \Phi \) and (7.1) for all \( k \in \{1, \ldots, \mu_l\} \) and all \( l \in \{1, \ldots, d\} \) we calculate
\[
(\text{diag}(\lambda_1, \ldots, \lambda_d) + \partial_{\mathfrak{e}_\mu}) \Phi N^{k-1}u_l = \lambda_l \frac{1}{(\mu_l-k)!}e_{d,l}z^{\mu_l-k} + \frac{1}{(\mu_l-(k+1))!}e_{d,l}z^{\mu_l-(k+1)}
\]
\[
= \lambda_l \Phi N^{k-1}u_l + \Phi N^Ku_l
\]
\[
= \Phi(\lambda_l N^{k-1}u_l + N^Ku_l)
\]
\[
= \Phi TN^{k-1}u_l.
\]

Therefore
\[
\Phi T = (\text{diag}(\lambda_1, \ldots, \lambda_d) + \partial_{\mathfrak{e}_\mu}) \Phi.
\]

The function defined by
\[
\Psi N^{k-1}w_l = e_{d,l}z^{k-1} \quad \text{for all} \quad k \in \{1, \ldots, \mu_l\} \quad \text{and} \quad l \in \{1, \ldots, d\}
\]
extends linearly to a linear bijection \( \Psi : \mathfrak{H} \to \mathfrak{C}^\mu_l \). The proof of the last equality is similar to the first part of this proof. \( \square \)

The operators \( \partial_{\mathfrak{e}_\mu} \) and \( Z_{\mathfrak{e}_\mu} \) were studied in [3, Section 8] in connection with operators without eigenvalues in finite-dimensional vector spaces.

References


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