The opening example in [2]

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1. Finding eigenvalues

Consider the following eigenvalue problem

$$-f''(x) = \lambda\,(\operatorname{sgn} x) f(x), \quad x \in [-1,1], \tag{1.1}$$

$$f'(1) = \lambda\, f(-1), \tag{1.2}$$

$$-f'(-1) = \lambda\, f(1). \tag{1.3}$$

With

$$\mathsf{b}(f) = \begin{bmatrix} f(-1) \\ f(1) \\ f'(-1) \\ f'(1) \end{bmatrix}.$$

and

$$\mathsf{M} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad \mathsf{N} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

the given eigenvalue problem can be written as

$$-f''(x) = \lambda\,(\operatorname{sgn} x) f(x), \quad x \in [-1,1],$$
$$\mathsf{M}\mathbf{b}(f) = \lambda\,\mathsf{N}\mathbf{b}(f).$$

The matrix Δ is calculated as

$$\Delta := -i\big(\mathsf{M}\mathsf{Q}^{-1}\mathsf{N}^*\big)^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It follows from [1, Remark 3.2] that the "corresponding positive definite" problem (for the meaning of this see [1]) is

$$-f''(x) = \lambda\, f(x), \quad x \in [-1,1],$$
$$\Delta\mathsf{M}\mathbf{b}(f) = \lambda\,\mathsf{N}\mathbf{b}(f),$$

that is

$$-f''(x) = \lambda f(x), \quad x \in [-1,1], \tag{1.4}$$

$$-f'(-1) = \lambda f(-1), \tag{1.5}$$

$$f'(1) = \lambda f(1). \tag{1.6}$$

A non-negative operator in a Hilbert space $L_2[-1,1] \oplus \mathbb{C}^2$ can be associated with the problem (1.4)-(1.6), see [1]. It follows from [1] that the operator associated with the problem (1.1)-(1.3) in the Krein space $L_{2,\mathrm{sgn}}[-1,1] \oplus \mathbb{C}^2_\Delta$ is a nonnegative definitizable operator. Hence it has real spectrum and Jordan chain can occur only at 0.

To find eigenvalues of the problem (1.1)-(1.3) we define the following two functions:

$$\mathrm{C}(x) := \begin{cases} \cosh(x) & \text{for } -1 \le x < 0, \\ \cos(x) & \text{for } \quad 0 \le x \le 1, \end{cases} \qquad \mathrm{S}(x) := \begin{cases} \sinh(x) & \text{for } -1 \le x < 0, \\ \sin(x) & \text{for } \quad 0 \le x \le 1. \end{cases}$$

1.1. The positive eigenvalues

To determine positive eigenvalues we set $\lambda = \mu^2, \mu > 0$. Then, the general solution of the equation

$$-f''(x) = \mu^2 \,(\operatorname{sgn} x) f(x), \quad x \in [-1,1],$$

is given by

$$a\ \mathrm{C}(\mu\, x) + b\ \mathrm{S}(\mu\, x), \quad x \in [-1,1],$$

where a and b are arbitrary complex numbers. Clearly

$$\mathbf{b}\big(a\ \mathrm{C}(\mu\,\cdot) + b\ \mathrm{S}(\mu\,\cdot)\big) = \begin{bmatrix} a\cosh(\mu) - b\sinh(\mu) \\ a\cos(\mu) + b\sin(\mu) \\ \mu\big(-a\sinh(\mu) + b\cosh(\mu)\big) \\ \mu\big(-a\sin(\mu) + b\cos(\mu)\big). \end{bmatrix}$$

A positive number $\lambda = \mu^2, \mu > 0$, is an eigenvalue of the given problem if and only if the system

$$\mathsf{M}\mathbf{b}\big(a\ \mathrm{C}(\mu\,\cdot) + b\ \mathrm{S}(\mu\,\cdot)\big) = \mu^2\, \mathsf{N}\mathbf{b}\big(a\ \mathrm{C}(\mu\,\cdot) + b\ \mathrm{S}(\mu\,\cdot)\big)$$

has a nontrivial solution for a and b. We rewrite this system in expanded form as

$$-a\,\mu\big(\mu\cosh(\mu) + \sin(\mu)\big) + b\,\mu\big(\cos(\mu) + \mu\sinh(\mu)\big) = 0$$
$$-a\,\mu\big(\mu\cos(\mu) - \sinh(\mu)\big) - b\,\mu\big(\cosh(\mu) + \mu\sin(\mu)\big) = 0.$$

The determinant of this system is

$$\mu^2\left(2\mu + \left(\mu^2+1\right)\cosh(\mu)\sin(\mu) + \left(\mu^2-1\right)\cos(\mu)\sinh(\mu)\right).$$

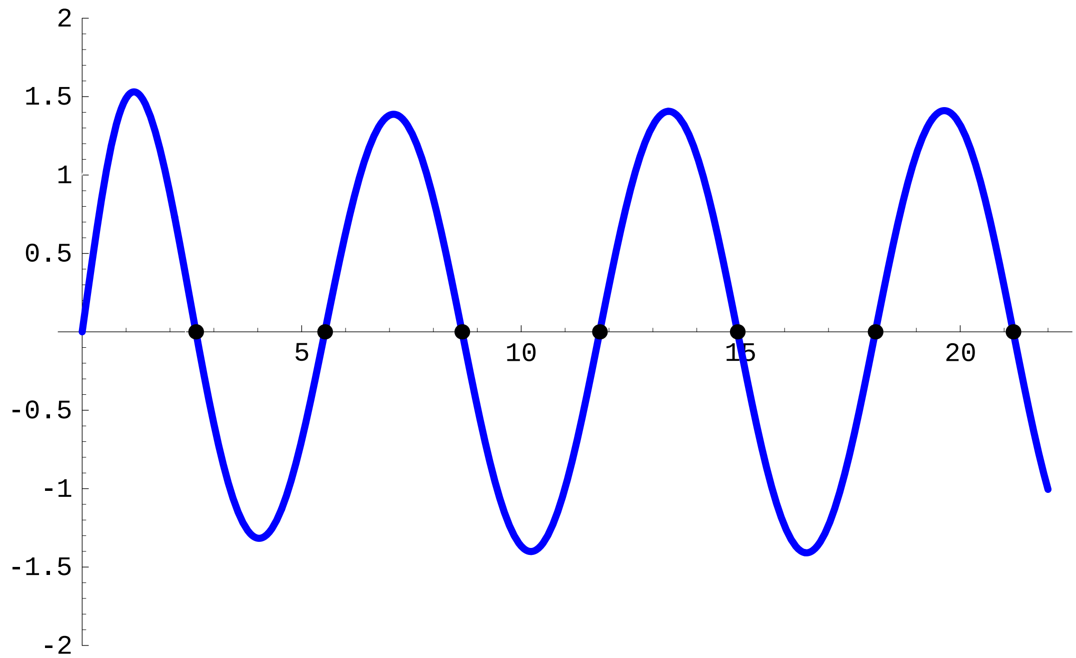

FIGURE 1. μ-s for positive eigenvalues

Therefore, the positive eigenvalues $\lambda = \mu^2, \mu > 0$, are determined from the positive solutions μ of the equation

$$\frac{2\mu}{\left(\mu^2+1\right)\cosh(\mu)} + \sin(\mu) + \frac{\left(\mu^2-1\right)}{\mu^2+1}\tanh(\mu)\cos(\mu) = 0. \tag{1.7}$$

A plot of this function is in Figure 1.

Approximate values for the first twelve solutions for μ are

$$\begin{array}{llll} 2.59044, & 5.52848, & 8.65279, & 11.7882, \\ 14.9271, & 18.0672, & 21.208, & 24.349, \\ 27.4903, & 30.6316, & 33.773, & 36.9144. \end{array}$$

All positive eigenvalues are simple since the equality

$$\mathsf{M}\mathbf{b}\big(\mathrm{S}(\mu\,\cdot)\big) = \mu^2\,\mathsf{N}\mathbf{b}\big(\mathrm{S}(\mu\,\cdot)\big)$$

does not hold for any $\mu > 0$. This is clear when written in the expended form and simplified:

$$\begin{bmatrix} \cos(\mu) \\ -\cosh(\mu) \end{bmatrix} = \begin{bmatrix} -\mu\sinh(\mu) \\ \mu\sin(\mu) \end{bmatrix}.$$

Thus, $\mathrm{S}(\mu\,\cdot)$ is not an eigenfunction. This implies that all positive eigenvalues are simple.

1.2. The eigenvalue 0

It is easy to see that 0 is the eigenvalue of this system and its geometric multiplicity is 1. Also, a nontrivial Jordan chain cannot exist. To justify this statement think of the given eigenvalue problem as the eigenvalue problem for the operator

$$\begin{bmatrix} f(x) \\ \mathsf{N}\mathbf{b}(f) \end{bmatrix} \quad \mapsto \quad \begin{bmatrix} -(\operatorname{sgn} x) f''(x) \\ \mathsf{M}\mathbf{b}(f) \end{bmatrix}.$$

Clearly the constant function 1 is an eigenfunction corresponding to 0 eigenvalue and

$$\mathsf{N}\mathbf{b}(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

To find a Jordan chain we have to find a function $g(x)$ such that

$$\begin{bmatrix} -(\operatorname{sgn} x) g''(x) \\ \mathsf{M}\mathbf{b}(g) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Since

$$\mathsf{M}\mathbf{b}(g) = \begin{bmatrix} g'(1) \\ -g'(-1) \end{bmatrix},$$

we have to find a function g such that $g''(x) = -\operatorname{sgn} x$ and $g'(1) = 1$ and $g'(-1) = -1$. As g'' is odd, g' must be even. Therefore $g'(1) = 1$ and $g'(-1) = -1$ is not possible. This proves that 0 is a simple eigenvalue.

Notice that in an earlier version (arXiv:0705.4157v2) of [2] we made a wrong statement that the algebraic multiplicity of 0 is 2.

1.3. The negative eigenvalues

To determine negative eigenvalues we set $\lambda = -\mu^2, \mu > 0$. Then, the general solution of the equation

$$-f''(x) = -\mu^2\,(\operatorname{sgn} x) f(x), \quad x \in [-1,1],$$

is given by

$$a\ \mathrm{C}(-\mu\, x) + b\ \mathrm{S}(-\mu\, x), \quad x \in [-1,1],$$

where a and b are arbitrary complex numbers. Clearly

$$\mathbf{b}\big(a\ \mathrm{C}(-\mu\,\cdot) + b\ \mathrm{S}(-\mu\,\cdot)\big) = \begin{bmatrix} a\cos(\mu) + b\sin(\mu) \\ a\cosh(\mu) - b\sinh(\mu) \\ \mu\big(a\sin(\mu) - b\cos(\mu)\big) \\ \mu\big(a\sinh(\mu) - b\cosh(\mu)\big). \end{bmatrix}$$

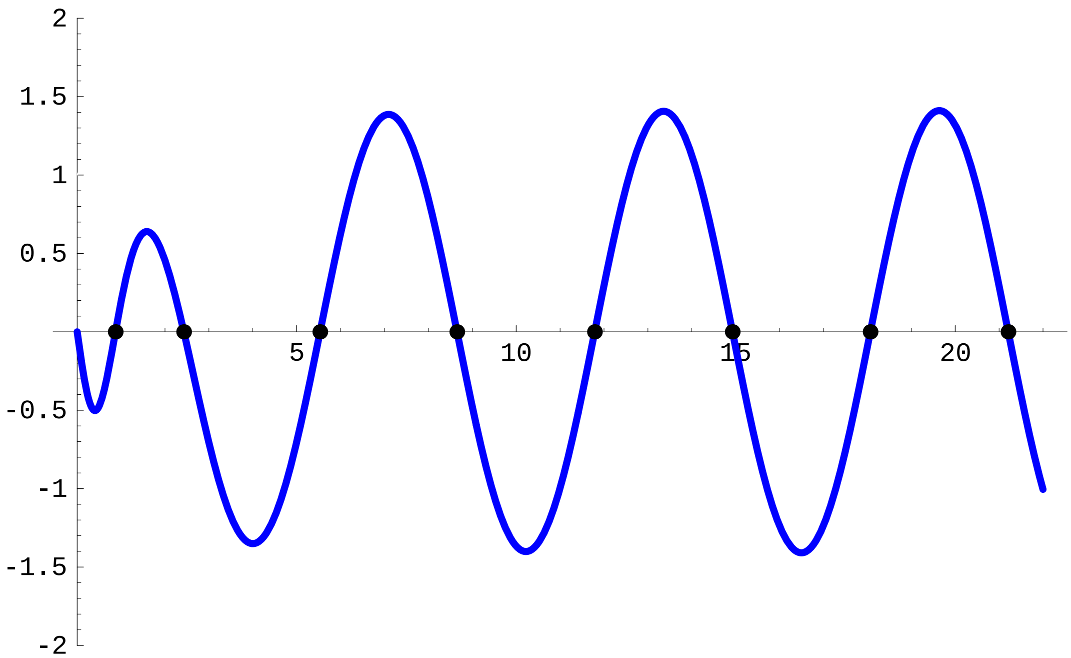

FIGURE 2. μ-s for negative eigenvalues

A negative number $\lambda = -\mu^2, \mu > 0$, is an eigenvalue of the given problem if and only if the system

$$\mathsf{M}\mathbf{b}\big(a\ \mathrm{C}(-\mu\,\cdot) + b\ \mathrm{S}(-\mu\,\cdot)\big) = -\mu^2\,\mathsf{N}\mathbf{b}\big(a\ \mathrm{C}(-\mu\,\cdot) + b\ \mathrm{S}(-\mu\,\cdot)\big)$$

has a nontrivial solution for a and b. We rewrite this system in expanded form as

$$\begin{aligned} a\,\mu\big(\mu\cos(\mu) + \sinh(\mu)\big) + b\,\mu\big(-\cosh(\mu) + \mu\sin(\mu)\big) &= 0 \\ a\,\mu\big(\mu\cosh(\mu) - \sin(\mu)\big) + b\,\mu\big(\cos(\mu) - \mu\sinh(\mu)\big) &= 0 \end{aligned}$$

The determinant of this system is

$$-\mu^2\left(-2\mu + \left(\mu^2+1\right)\cosh(\mu)\sin(\mu) + \left(\mu^2-1\right)\cos(\mu)\sinh(\mu)\right).$$

Therefore, the negative eigenvalues $\lambda = -\mu^2, \mu > 0$, are determined from the positive solutions μ of the equation

$$-\frac{2\mu}{\left(\mu^2+1\right)\cosh(\mu)} + \sin(\mu) + \frac{\left(\mu^2-1\right)}{\mu^2+1}\tanh(\mu)\cos(\mu) = 0. \tag{1.8}$$

A plot of this function is in Figure 2.

Approximate values for the first twelve solutions for μ are

0.872674,	2.43031,	5.53248,	8.65268,
11.7882,	14.9271,	18.0672,	21.208,
24.349,	27.4903,	30.6316,	33.773.

All negative eigenvalues are simple since the equality

$$\mathsf{M}\mathbf{b}\big(\mathrm{S}(-\mu\,\cdot)\big) = -\mu^2\,\mathsf{N}\mathbf{b}\big(-\mathrm{S}(\mu\,\cdot)\big)$$

does not hold for any $\mu > 0$. This is clear when written in the expended form and simplified:

$$\begin{bmatrix} -\cosh(\mu) \\ \cos(\mu) \end{bmatrix} = \begin{bmatrix} -\mu\sin(\mu) \\ \mu\sinh(\mu) \end{bmatrix}.$$

Thus, $\mathrm{S}(-\mu\,\cdot)$ is not an eigenfunction. This implies that all negative eigenvalues are simple.

2. Eigenfunctions

The eigenfunctions corresponding to positive eigenvalues are obtained when the positive solutions of (1.7) are substituted in

$$\mathrm{C}(\mu\,x)\left(\mu\tanh(\mu) + \frac{\cos(\mu)}{\cosh(\mu)}\right) + \mathrm{S}(\mu\,x)\left(\mu + \frac{\sin(\mu)}{\cosh(\mu)}\right).$$

Using the approximate values for μ we give plots of eigenfunctions corresponding to the first 12 positive eigenvalues in Table 1.

The eigenfunctions corresponding to negative eigenvalues are obtained when the positive solutions of (1.8) are substituted in

$$\mathrm{C}(-\mu\,x)\left(1 - \frac{\mu\sin(\mu)}{\cosh(\mu)}\right) + \mathrm{S}(-\mu\,x)\left(\tanh(\mu) + \frac{\mu\cos(\mu)}{\cosh(\mu)}\right)$$

Using the approximate values for μ we give plots of eigenfunctions corresponding to the first 12 negative eigenvalues in Table 2.

3. Eigenfunction expansions

This section will be added later.

References

[1] P. Binding, B. Ćurgus, Form domains and eigenfunction expansions for differential equations with eigenparameter dependent boundary conditions. Canad. J. Math. **5**4 (2002), 1142–1164. (2004), 244–248.

[2] P. Binding, B. Ćurgus, Riesz basis of root vectors of indefinite Sturm-Liouville problems with eigenparameter dependent boundary conditions. II. Submitted for publication.

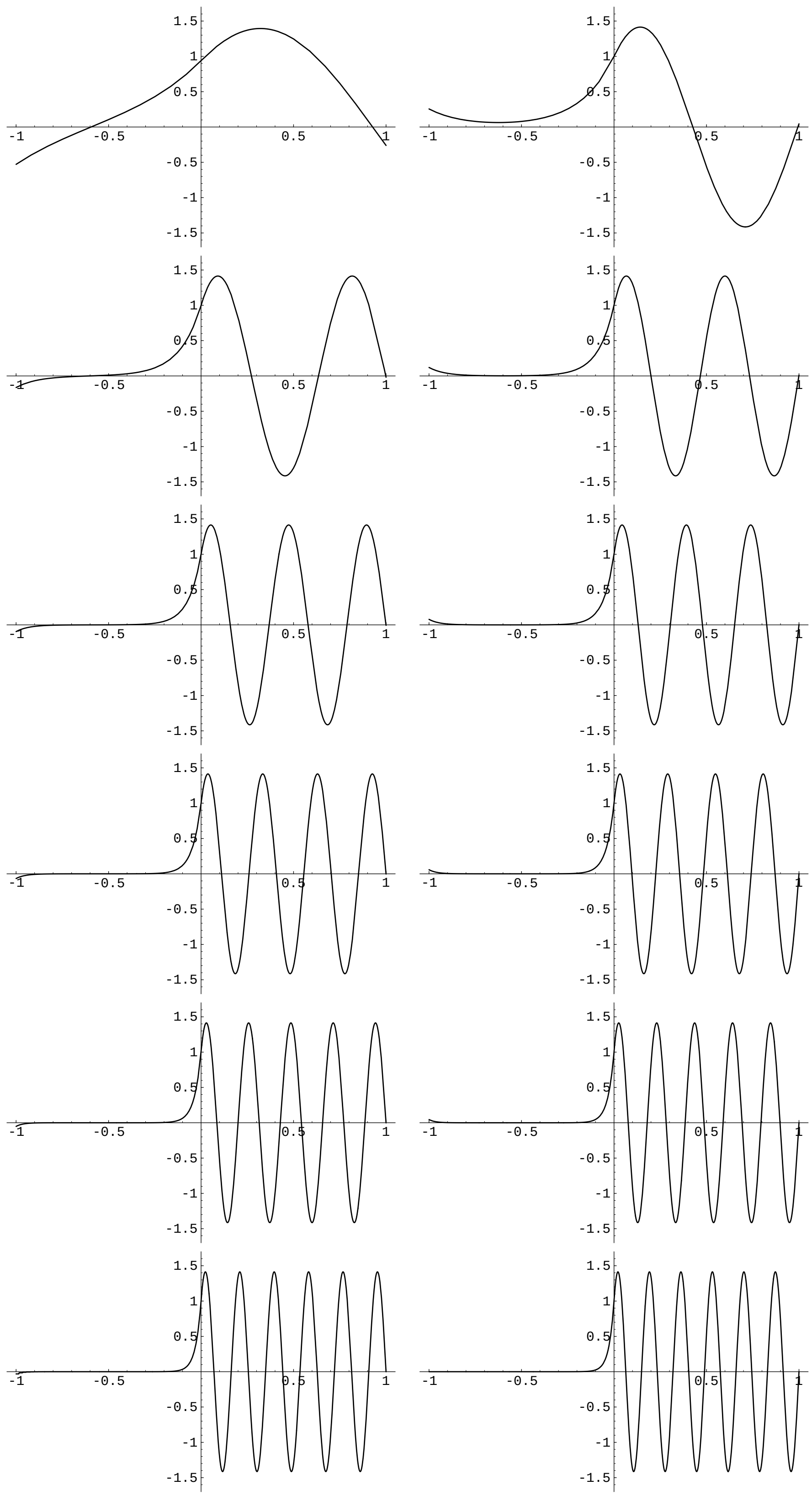

Table 1. Eigenfunctions corresponding to the first 12 positive eigenvalues

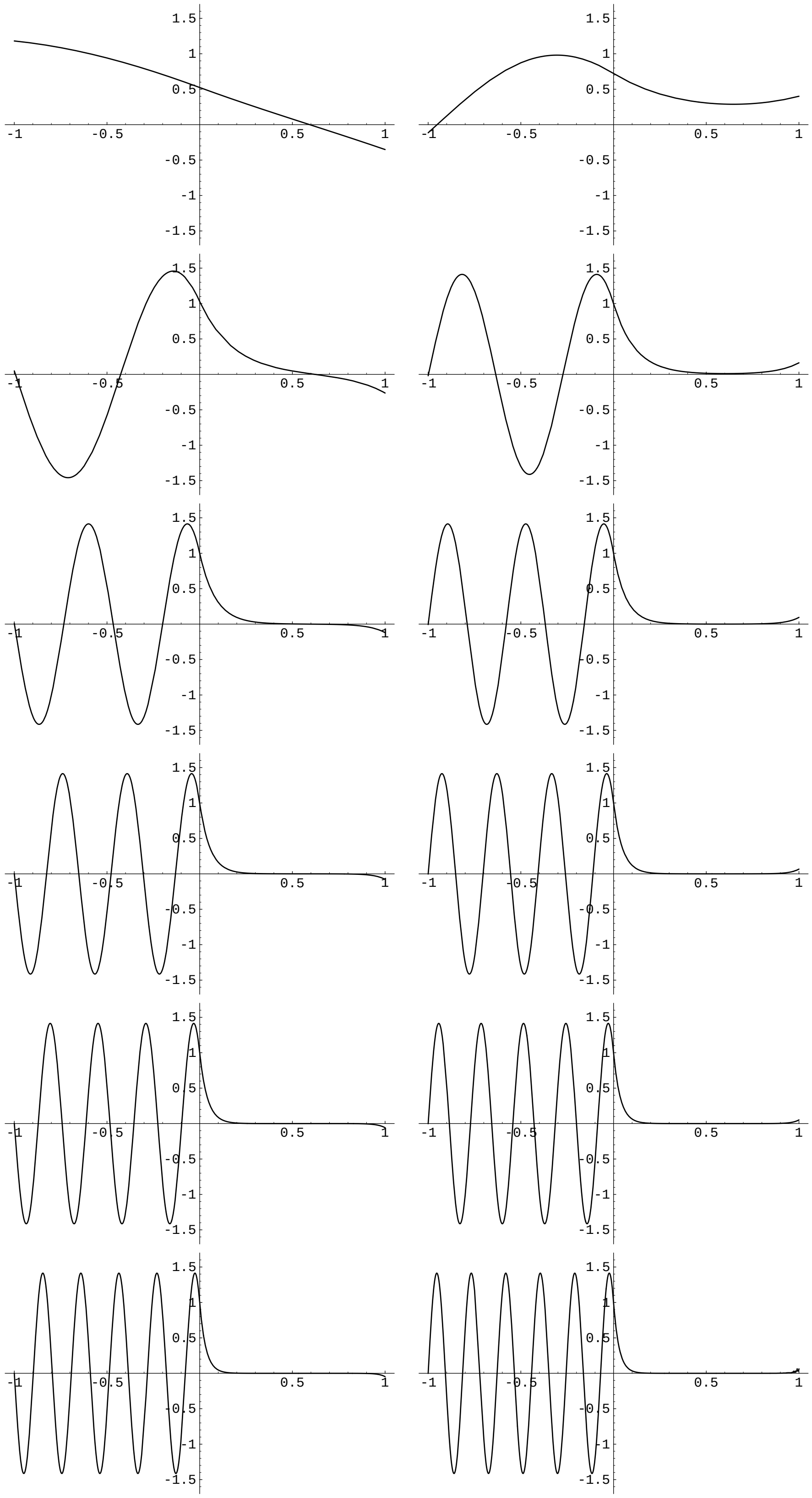

Table 2. Eigenfunctions corresponding to the first 12 negative eigenvalues

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