of zeros forcing P to have likewise a bottom row of zeros, and this contradicts the invertibility of P. Thus $H = I$, $C = P$, and the equation $PA = H$ is actually $CA = I$.

This argument shows at once that (i) a matrix is invertible if and only if its reduced row echelon form is the identity matrix, and (ii) the set of invertible matrices is precisely the set of products of elementary matrices.

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Root Preserving Transformations of Polynomials

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Consider the real vector space $\mathcal{P}_2$ of all polynomials of degree at most 2. High-school students study the roots of the polynomials in $\mathcal{P}_2$, while linear algebra students study linear transformations on $\mathcal{P}_2$. Is it possible to bring these two groups together to do some joint research?

For example, a linear algebra student chooses a specific linear transformation $T : \mathcal{P}_2 \to \mathcal{P}_2$ and asks others to study the roots of a polynomial

$$p(x) = ax^2 + bx + c, \quad x \in \mathbb{R},$$

and the roots of its image

$$(Tp)(x) = bx^2 + cx + a, \quad x \in \mathbb{R}. \tag{1}$$

Here a, b, and c are arbitrary real numbers. The students may immediately notice that the polynomial $x^2 + x + 1$ is unchanged by this transformation. Hence this particular polynomial and its image have the same (complex) roots. After some "trial and error," a high-school student points out that the polynomial $x^2 + 3x + 2$ has the roots -1 and -2, while its image $3x^2 + 2x + 1$ does not have real roots. Their next interesting discovery is that, with $v \neq 1$, the polynomial $x^2 + (v-1)x - v$ has roots 1 and $-v$, while its image $(v-1)x^2 - vx + 1$ has roots 1 and $1/(v-1)$. This is curious since in this case a polynomial and its image have one common root, namely 1.

After further study the students conclude that there doesn't seem to be any general simple relationship between the roots of a polynomial p and the roots of its image Tp under the linear transformation given by (1). But the obvious fact is that there are plenty of other linear transformations on $\mathcal{P}_2$; will it always be the case that there is no simple relationship between the roots? Clearly, a non-zero multiple of the identity on

$\mathcal{P}_2$ does not change the roots of a polynomial, at all, and so such linear transformations are of no big interest in this study.

In the rest of the note, instead of $\mathcal{P}_2$, we consider the (complex or real) vector space $\mathcal{P}_n$ of all polynomials of degree at most n. To cover both cases, $\mathbb{F}$ stands for $\mathbb{R}$ if we consider $\mathcal{P}_n$ as a real vector space and $\mathbb{F}$ stands for $\mathbb{C}$ if we consider $\mathcal{P}_n$ as a complex vector space. For p in $\mathcal{P}_n$ we denote by $Z(p)$ the set of all roots of p in $\mathbb{F}$.

Inspired by the students' investigations we ask the following question:

Is there a (non-trivial) linear transformation T from $\mathcal{P}_n$ to $\mathcal{P}_n$ such that for each $p \in \mathcal{P}_n$ with a root in $\mathbb{F}$, the polynomials p and Tp have a common root?

Surprisingly, it seems that this question has not been addressed in the literature. The first author of this note has been assigning it at various levels of linear algebra courses. His experience is that students find it quite challenging even in the case $n = 2$. Students often offer "brute force" proofs that are based on calculating the matrix for the transformation T entry by entry.

In the next theorem we give a general answer to the above question. In the proof we use only elementary linear algebra and Taylor polynomials.

THEOREM. *Let $T \neq 0$ be a linear transformation from $\mathcal{P}_n$ to $\mathcal{P}_n$. Then*

$$Z(p) \cap Z(Tp) \neq \emptyset \quad \textit{for all} \quad p \in \mathcal{P}_n \quad \textit{such that} \quad Z(p) \neq \emptyset \tag{2}$$

if and only if T is a non-zero multiple of the identity on $\mathcal{P}_n$.

Proof. The "if" part of the theorem is obvious. To prove the "only if" part we assume (2).

Let $p \in \mathcal{P}_n$ be arbitrary. To prove that Tp is a constant multiple of p we choose an arbitrary $w \in \mathbb{F}$ and evaluate $(Tp)(w)$. To this end we consider the following $n+1$ polynomials in $\mathcal{P}_n$

$$e_0(x) := 1, \quad e_{k,w}(x) := (x-w)^k, \quad x \in \mathbb{F}, \quad k = 1, \ldots, n. \tag{3}$$

With notation (3), the nth degree Taylor polynomial of p at w is

$$p(x) = p(w)e_0(x) + \sum_{k=1}^{n} \frac{p^{(k)}(w)}{k!} e_{k,w}(x), \quad x \in \mathbb{F}.$$

This equality provides a representation of p as a linear combination of the polynomials in (3). Applying T to both sides of the last equality and using the linearity of T we obtain

$$(Tp)(x) = p(w)(Te_0)(x) + \sum_{k=1}^{n} \frac{p^{(k)}(w)}{k!} (Te_{k,w})(x), \quad x \in \mathbb{F}. \tag{4}$$

Clearly, $Z(e_{k,w}) = \{w\} \neq \emptyset$ for all $k = 1, \ldots, n$. Therefore, by assumption (2),

$$\emptyset \neq Z(e_{k,w}) \cap Z(Te_{k,w}) = \{w\} \cap Z(Te_{k,w}).$$

Consequently, $w \in Z(Te_{k,w})$ and thus

$$(Te_{k,w})(w) = 0 \quad \text{for all} \quad k = 1, \ldots, n.$$

Now we set $x = w$ in (4) and use the preceding n equalities to get

$$(Tp)(w) = p(w)(Te_0)(w). \tag{5}$$

Notice that $w \in \mathbb{F}$ and $p \in \mathcal{P}_n$ in (5) are arbitrary. Since the degree of the polynomial $Tp \in \mathcal{P}_n$ is less than or equal to n, if we choose $p \in \mathcal{P}_n$ to be of degree n, then (5) implies that the degree of Te_0 must be zero. That is, Te_0 is a constant polynomial: $(Te_0)(w) = c$ for all $w \in \mathbb{F}$, and so (5) implies that T is a multiple of the identity. ∎

Now, the next natural (but quite a bit harder) question would be the following:

Characterize those linear transformations T from $\mathcal{P}_n$ to $\mathcal{P}_n$ such that, for some constant $C > 0$ and for all $p \in \mathcal{P}_n$ with $Z(p) \neq \emptyset$, some zeros of polynomials p and Tp are at most "distance C apart."

The notion itself of distance between the zero sets $Z(p)$ and $Z(Tp)$ needs to be clarified, of course, but this question has also been completely answered by the authors and the results will appear in a forthcoming article [**1**]. A similar question was also considered in [**2**].

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Do Cyclic Polygons Make the Cut?

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For thousands of years mathematicians have studied the properties of *cyclic polygons*—polygons that can be circumscribed by a circle. There also exists a large number of findings concerning *cyclic product relations* for polygons—products of ratios of segment lengths, as in [**2**] and the theorem of Menelaus (see FIGURE 1). This paper intends to mix the two topics together, offering results reminiscent of but distinct from those found in [**4**] and [**6**]. A product of length-ratios is the primary focus, but rather than dealing with a single polygon we look at the interaction between a pair of cyclic polygons.

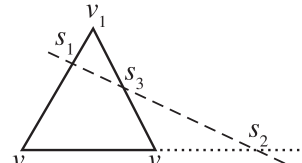

Figure 1 Menelaus' theorem states $\frac{|v_1 s_1|}{|s_1 v_2|} \cdot \frac{|v_2 s_2|}{|s_2 v_3|} \cdot \frac{|v_3 s_3|}{|s_3 v_1|} = 1$

On a circle we find the vertices of a polygon $V = [v_1, v_2, \ldots, v_n]$, where the order of this set indicates the connection of the vertices. Instead of cutting this