Mathematical Expectation

Properties of Mathematical Expectation I

The concept of mathematical expectation arose in connection with games of chance. In its simplest form, mathematical expectation is the product of the amount a player stands to win and the probability that the player would win. For example, if one of the 1,000 tickets of a raffle paid a prize of $5,000, the mathematical expectation of the lottery would be .001×$5,000 = $5. This figure is indeed the average; altogether the 1000 tickets pay $5,000 so the average ticket is worth $5,000/1000 = $5. It is possible to alter this example slightly to demonstrate mathematical expectation of a more complicated lottery. For instance, if two second place tickets paid the holders $1,000 each, then the mathematical expectation of the lottery is .001×$5,000+.002×$1,000=$7.

In the preceding example, the amount won in the lottery was a random variable, and the mathematical expectation of this random variable was the sum of the products obtained by each value of the random variable by the corresponding probability. Referring to the mathematical expectation of a random variable simply as its expected value, we have the following two definitions:

Definition #1

If X is a discrete random variable and f(x) is the value of its probability distribution at x, the expected value of X is

$$E[X] = \sum_{x} x f(x)$$

If X is a continuous random variable and f(x) is the value of its probability distribution at x, the expected value of X is

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Example #1

A casino is considering a dice game that would pay the winner of the game $10. The game is similar to craps, the participant would roll two fair, 6-sided dice and if they sum to 7 or 11, the participant wins; otherwise they lose. What is the expected payout the casino will make as each game is played?

One first needs to identify the probability distribution f(x) of the sum of two dice. Below is a table that identifies these probabilities:

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$1/36$	$1/18$	$1/12$	$1/9$	$5/36$	$1/6$	$5/36$	$1/9$	$1/12$	$1/18$	$1/36$

Since the casino stands to lose $10 each time a contestant roles a 7 or 11, the mathematical expected value (or expected cost) of this game to the casino is:

$$1/36 \times 0 + 1/18 \times 0 + 1/12 \times 0 + 1/9 \times 0 + 5/36 \times 0 + 1/6 \times 10 + 5/36 \times 0 + 1/9 \times 0 + 1/12 \times 0 +$$

$\frac{1}{18}\times 10 + \frac{1}{36}\times 0$ = \$2.22. The expected value of this game is \$2.22.

Problems for Consideration:

1. A lot of 12 Beethoven CDs includes two with the composer's autograph. If three CDs are chosen for random for shipment to radio stations, how many CDs with autographs can the shipper expect to send to any radio station?

2. Managers of an airline know that .5% of an airline's passengers lose their luggage on domestic flights. Management also knows that the average value claimed for a lost piece of luggage on domestic flights is \$600. The company is considering increasing fares by an appropriate amount to cover expected compensation to passengers who lose their luggage. By how much should the airline increase fares?

3. If X is the number of points rolled with a balanced die, and I pay you according to the formula $2x^2 + 1$, what is the expect value of pay you receive after rolling one die?

Properties of Mathematical Expectation II

Linearity: If X and Y are random variables and a and b are constants, then:

i $$E[aX]=aE[X]$$

ii $$E[X + b] = E[X] + b$$

The first property suggests that the mathematical expectation of a constant times a random variable is equal to the expectation of a random variable times a constant. In example #1, one may ask, what is the expected value of playing this dice game 20 times? Statement i suggests that the expected value is simply 20×\$2.22 = \$44.40. Statement ii implies that the expectation of a random variable plus a constant is equal to the constant plus the expectation of the random variable.

Problems for Consideration:

4. What is the expected value of a game that works as follows: I flip a coin and, if tails pay you \$2; if heads pay you \$1. In either case I also pay you \$.50.

Properties of Mathematical Expectation III

The mathematical expectation of a constant is that constant:

iii $$E[a]=a$$

If X and Y are independent random variables then

iv $$E[XY] = E[X]\times E[Y]$$

Problems for Consideration:

5. I offer two simultaneous games. One game, is explained in problem #4. The other game is a game where I roll a six sided die and pay you the number of dollars shown on the face of the die. Both the coin and the die are determined simultaneously and are independent of each other. I will pay you the product of the two resulting games. What is the expected amount that I pay in this joint game?

Properties of Mathematical Expectation IV

Upon considering expectation, I hope that its similarity to "averages" strikes you. This suggests that expectations have application to any statistical process involving means or averages. Perhaps the most commonly applied mathematical expectations is to variances. Consider the formula for variance: $\operatorname{Var}(X)=\frac{\sum\left(X_i-\overline{X}\right)^2}{n}$. One way to read this formula is that the variance of X is the average of the sum of the squared difference between X and its mean. Since $\overline{X}$ is the expected value of X, one way to write this formula is $\operatorname{Var}(X)=\frac{\sum(X_i-E[X])^2}{n}$. Of course, since the expected value of a random variable is its average, a better way to write this formula is $\operatorname{Var}(X)=E[X-E[X]]^2$.

The expectations operator is a linear operator that follows all of the order of algebraic operations. Thus, we can expand the variance expression in the following manner (using FOIL) :

$$\operatorname{Var}(X)=E[X-E[X]]^2=E\left[X^2-2XE[X]+E[X]^2\right]=E\left[X^2\right]-2E[X[E[X]]]+E[X]^2$$

$$=E\left[X^2\right]-2E[X]^2+E[X]^2=E\left[X^2\right]-E[X]^2$$

The variance of a random variable is simple the expectation of its square minus the square of its expectation. To show this I used what is known as the rule of iterative expectations—that is the expectation of an expectation of X is simply the expectation of X.

Example #2

Consider the case of flipping a fair coin and paying \$1 if heads and \$2 if tails. It is clear that the expected value of this activity is \$1.5. Using the $\operatorname{Var}(X)=\frac{\sum\left(X_i-\overline{X}\right)^2}{n}$ formula it is also clear that the variance of this game is $\frac{(1-1.5)^2+(2-1.5)^2}{2}=.25$. Using the fact that $\operatorname{Var}[X]=E\left[X^2\right]-E[X]^2$ gives the following computations: the squared random variables are 1^2 and 2^2 and since each has a 50% probability of being observed, the $E\left[X^2\right]=\frac{1}{2}\times 1+\frac{1}{2}\times 4=2.5$. Since the expected value of this game is 1.5, it must be that $E[X]^2=1.5^2=2.25$. Thus, the $\operatorname{Var}[X] = 2.5 - 2.25 = .25$.

Problems for Consideration:

6. Use the technique demonstrated in Example #2 to find the variance of a fair, six-sided die.

7. Prove that if $E[X] = 0$ then $\operatorname{Var}[X] = E[X^2]$.

Properties of Mathematical Expectation V

As shown above, the variance of a random variable is simply and extension of mathematical expectations. Using the fact that Var[X] = $E[X-E[X]]^2$, it is possible to show a number of common facts that students encounter in statistics. For instance, if and b and are constants, then

v $\quad\quad Var[aX]=a^2Var[X]$

vi $\quad\quad Var[X + b] = Var[X]$

Problems for Consideration:

8. Prove statements v and vi.
9. Let X be a random variable and Y = 2X + 1. What is the variance of Y?

Properties of Mathematical Expectation VI

One common use of mathematical expectations is the covariance between two independent variables. The covariance is traditionally defined as $\mathrm{Cov}(X,Y)=\frac{\sum\left((X_i-\overline{X})(Y_i-\overline{Y})\right)}{n}$, which, when you stop to think about it, is simply the average of the sum of the product of the deviation from the mean of X and Y or $\mathrm{Cov}(X,Y)=E[(X-E[X])(Y-E[Y])]$. Using the same technique in section IV, it can be shown that $\mathrm{Cov}(X,Y)=E[XY]-E[X]\times E[Y]$.

Example #3

It should be clear that throwing a fair die and a fair coin should be independent and thus have no covariance. Let heads (X) represent the number 1 and tails the number 2—it is clear that E[X]=1.5. The mathematical expectation of a fair die (Y) is 3.5—thus, E[Y]=3.5. What is E[XY]? Consider the following table of probabilities:

	X = 1	X = 2
Y =1	XY = 1, prob = $1/12$	XY = 2, prob = $1/12$
Y =2	XY = 2, prob = $1/12$	XY = 4, prob = $1/12$
Y =3	XY = 3, prob = $1/12$	XY = 6, prob = $1/12$
Y =4	XY = 4, prob = $1/12$	XY = 8, prob = $1/12$
Y =5	XY = 5, prob = $1/12$	XY = 10, prob = $1/12$
Y =6	XY = 6, prob = $1/12$	XY = 12, prob = $1/12$

The E[XY] is equal to $1/12\,(1+2+3+4+5+6+2+4+6+8+10+12)=5.25$. The Cov[XY] is thus 5.25-1.5×3.5 = 0—exactly what was expected.

Problems for Consideration:

10. Prove that if X and Y are independent then E[XY] = E[X]×E[Y].
11. Many times I have needed to create two random variables that are correlated with one another. To do this, I typically have the computer generate a random variable X that is normally distributed with E[X] = 0 and Var[X] = 1. I then create a second independent variable ε that is also normally distributed with E[ε]=0 and Var[ε] = 1. Finally, I create the variable that is correlated with Y by setting Y = X + aε where a is some parameter I choose in order to control

the correlation between X and Y. What value of a do I need to choose to find a correlation between X and Y of .5. What happens to this correlation as a gets larger?

Answers to Problems for Consideration

1. With some work, you should be able to determine that the probability of receiving 0, 1 or 2 signed copies of a CD when 3 are shipped out of 12 total is[1]

Signed CDs	0	1	2
probability	$6/11$	$9/22$	$1/22$

The expected number of CD's received is thus $6/11 \times 0 + 9/22 \times 1 + 1/22 \times 2 = .5$

2. $.005 \times \$600 = \$3$.

3.

X	1	2	3	4	5	6
$2X^2+1$	3	9	19	33	51	73
probability	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$E\left[2X^2+1\right] = 1/6\left(3+9+19+33+51+73\right) = 31\,1/3$.

4. By ii, the expectation of this game is \$2 (the expectation of the die part of the game is \$1.50 and I add to that a constant \$.50). To check:

Die Roll	Head	Tail
Payout	\$1.50	\$2.50
probability	$1/2$	$1/2$

The expectations of this payout is .5(1.5+2.5) = 2.

5. By iv the answer to this should be the product of the two expectations. As seen above, the expectation of the coin flip is 2. The expectation of a fair die roll is 3.5. Since these are independent events, my expected payout is $2\times3.5=7$. To check:

	Heads, X = 1 + .50	Tails, X = 2 + .50
Die Roll, Y =1	XY = 1.50, prob = $1/12$	XY = 2.50, prob = $1/12$
Die Roll, Y =2	XY = 3, prob = $1/12$	XY = 5, prob = $1/12$
Die Roll, Y =3	XY = 4.50, prob = $1/12$	XY = 7.50, prob = $1/12$
Die Roll, Y =4	XY = 6, prob = $1/12$	XY = 10, prob = $1/12$
Die Roll, Y =5	XY = 7.50, prob = $1/12$	XY = 12.50, prob = $1/12$
Die Roll, Y =6	XY = 9, prob = $1/12$	XY = 15, prob = $1/12$

[1] The probabilities are given by $f(x) = \dfrac{\binom{2}{x}\binom{10}{3-x}}{\binom{12}{3}}$ for x = 0,1,2.

The E[XY] is equal to $\frac{1}{12}(1.50+3+4.50+6+7.50+9+2.50+5+7.50+10+12.50+15)=7$, exactly as predicted!

6. The variance of a fair, six sided die is equal to $E[X^2]-E[X]^2$ where X is the result of a six sided die. The $E[X^2]$ is nothing more than the expectation (or average) of square of the die rolls. Thus, $E[X^2]$ is $\frac{1}{6}(1+4+9+16+25+36)=15\frac{1}{6}$. The average of a six sided die is 3.5 so the variance is equal to $15\frac{1}{6}-3.5^2 = 2.916$. To check, the variance of a six sided die is calculated as:

$$\frac{(1-3.5)^2+(2-3.5)^2+(3-3.5)^2+(4-3.5)^2+(5-3.5)^2+(6-3.5)^2}{6}=2.916.$$

7. $Var[X]=E[X-E[X]]^2 = E[X^2 - 2X\times E[X]+E[X]^2] = E[X^2] - 2E[X]\times E[X]+E[X]^2 = E[X^2] - E[X]^2$. If the E[X]=0, then $Var[X] = E[X^2] - 0 = E[X^2]$.

8. $Var[aX]=E[aX-E[aX]]^2 = E[aX-aE[X]]^2 = a^2[X-E[X]]^2 = a^2Var[X]$.

9. Var[Y] = Var[2X + 1] = 4Var[X] by v and vi. Here's a check:

$$Var(2X+1) = E\Big[\big[2X+1-E[2X+1]\big]\big[2x+1-E[2x+1]\big]\Big]$$

$$= E\Big[4x^2+2x-2xE[2x+1]+2x+1-E[2x+1] \quad 2xE[2x+1] \quad E[2x+1] + (E[2x+1])^2\Big]$$

$$= \cancel{4E[x^2]} + \cancel{2E[x]} - \cancel{4E[x]} - \cancel{2E[x]} + \cancel{2E[x]} + \cancel{E[1]} - 2E[x] - \cancel{2E[x]} - 2E[x] \; \cancel{E[1]} + 4E[x^2] - \cancel{E[1]} - \cancel{4E[x^2]} + \cancel{E[1]} + \cancel{4E[x]}$$

$$= 4E[x^2] - 4E[x] = 4\big[E[x^2] - E[x]\big] = 4\,Var(x)$$

10. Independence requires the Cov(X,Y) = 0. If Cov(X,Y) = 0 then $Cov(X,Y)=E[(X-E[X])(Y-E[Y])]=E[XY]-E[X]\times E[Y]=0$ If $E[XY]-E[X]\times E[Y]=0$ then it is necessary that $E[XY] = E[X]\times E[Y]$.

11. Here's my work:

$Y = X + a\,\mathcal{E}$

$E[\mathcal{E}] = 0 \quad Var[\mathcal{E}] = 1$

$E[X] = 0 \quad Var[X] = 1$

$$Cov(X,Y) = E\Big[(X - E(X))(Y - E(Y))\Big] =$$

$$= E\Big[(X - E(X))(X + a\mathcal{E} - E(X + a\mathcal{E}))\Big]$$

$$E\big[(X)(X + a\mathcal{E})\big] \qquad \text{b/c } E[X + a\mathcal{E}] = E[X] = 0$$

$$= E\big[X^2 + aX\mathcal{E}\big]$$

$$= E[X^2] + aE[X\mathcal{E}] \qquad \text{if } Cov(X,\mathcal{E}) = 0 \text{ then } E[X,\mathcal{E}] = 0$$

so $Cov(X,Y) = E[X^2]$ since $Var(X) = E(X^2) - (E[X])^2$ and $E[\ = 0$

$$Cov(X,Y) = Var(X) = 1$$

$$Correlation(X,Y) = \frac{Covariance(X,Y)}{\sqrt{Var(X)}\ \sqrt{Var(Y)}}$$

$$Var(Y) = E\Big[X + a\mathcal{E} - E[X + a\mathcal{E}]\Big]^2 = E\Big[X + a\mathcal{E} - E[X] - aE[\mathcal{E}]\Big]^2$$

$E[X] = 0 \quad E[\mathcal{E}] = 0$

$$= E\big[X + a\mathcal{E}\big]^2 = E[X^2] + a^2E[\mathcal{E}^2] + 2aE[\mathcal{E}X]$$

$E[\mathcal{E}X] = 0$ if $cov(\mathcal{E},X) = 0$ (which it is if $\mathcal{E}$ & X are independent).

$$Var(Y) = E[X^2] + a^2E[\mathcal{E}^2] = Var(X) + a^2\,Var(\mathcal{E})$$

since $E[X] = 0$ and $E[\mathcal{E}] = 0$

$$Correlation(X,Y) = \frac{1}{\sqrt{1}\ \sqrt{1 + a^2}}$$

If $correlation(X,Y) = 5$ then $a = \sqrt{3}$