A Structure Theorem for $\mathbb{P}^1 - \operatorname{Spec} k$-bimodules

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Our main result concerns the structure of objects in $\mathsf{Bimod}_k(\mathsf{Qcoh}\mathbb{P}^1_k, \mathsf{Mod}k)$ when $k = \overline{k}$.

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Theorem (Eilenberg, Watts 1960)

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Still true if $\text{Mod}S$ is replaced by $\text{Qcoh}\,Y$ where Y is a scheme.

Y scheme

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Integral Transforms in Algebraic Geometry

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When is $F \in \mathrm{Bimod}_k(\mathrm{Qcoh} X, \mathrm{Qcoh} Y)$ an integral transform?

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Non-commutative Space := Grothendieck Category

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- Proj $A := \mathrm{Gr}A/\mathrm{Tors}A$ where A is $\mathbb{Z}$-graded

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If $f : Y \to X$ is a morphism of schemes, (f^*, f_*) is an adjoint pair.

Adjoint functor theorem $\Rightarrow$

$$\text{Morphisms } f : Y \to Z \quad \leftrightarrow \quad \operatorname{Bimod}_k(\operatorname{Mod} Z, \operatorname{Mod} Y).$$

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Let $f : Y \to X$ denote a morphism of schemes such that $(f^*, f_*, f^!)$ is an adjoint triple (e.g. a closed immersion of varieties).

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Then
$$\mathrm{Qcoh}\, Y \underset{f^*}{\overset{f_*}{\rightleftharpoons}} \mathrm{Qcoh} X$$
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The latter may not come from a morphism of schemes.

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The Eilenberg-Watts Theorem over Schemes I: The Eilenberg-Watts Functor

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Question

Given $F \in \text{Bimod}_k(\text{Qcoh}X, \text{Qcoh}\,Y)$, which integral transform is closest to F?

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for some $\mathcal{F}_V \in \text{Qcoh}\,V \times Y$.

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Theorem (Van den Bergh, N.)

The collection $\mathcal{F}_V$ induces (via gluing) a functor

$$W(-) : \mathrm{Bimod}_k(\mathrm{Qcoh} X, \mathrm{Qcoh} Y) \to \mathrm{Qcoh} X \times Y$$

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$$F \cong \bigoplus_{i=m}^{\infty} \text{H}^1(\mathbb{P}^1, (-)(i))^{\oplus n_i}$$

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If $Y = \operatorname{Spec} k$:

- Projection $\tau : X \times Y \to X$ is isomorphism
- write $W(F)$ instead of $\tau_* W(F)$ and
- identify $\operatorname{Qcoh}(\operatorname{Spec} k)$ and $\operatorname{Mod} k$, so

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- Projection $\tau : X \times Y \to X$ is isomorphism
- write $W(F)$ instead of $\tau_* W(F)$ and
- identify $\operatorname{Qcoh}(\operatorname{Spec} k)$ and $\operatorname{Mod} k$, so

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$$F \cong \oplus_{i=-n}^{\infty} H^1(\mathbb{P}^1, (-)(i))^{\oplus n_i} \oplus H^0(\mathbb{P}^1, - \otimes W(F)).$$

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Exercise

Use main result to give elementary proof of Serre Duality, i.e. $\text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(-, \mathcal{O})^* \cong H^1(\mathbb{P}^1, - \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}(-2))$.

Overview of Proof of Main Theorem

Eilenberg-Watts over Schemes $\Rightarrow \exists$

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- $m > 0$. In this case a contradiction is found.

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Proposition

If $m > 0$, there exists $q \in \mathbb{P}^1$ such that $R_q \subset \ker \Gamma_F$.

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Then $\exists!$ natural transformation $\Omega : F \to G$ extending $\underline{\Omega}$.

An Open Question

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Thank you for your attention!