## A Structure Theorem for $\mathbb{P}^1$ – Spec *k*-bimodules

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## Conventions and Notation

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Our main result concerns the structure of objects in  $Bimod_k(Qcoh \mathbb{P}^1_k, Modk)$  when  $k = \overline{k}$ .

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### Theorem (Eilenberg, Watts 1960)

Every  $F \in \text{Bimod}_k(\text{Mod}R, \text{Mod}S)$  is an integral transform.

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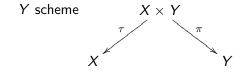
Still true if ModS is replaced by QcohY where Y is a scheme.

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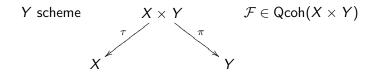
## Y scheme

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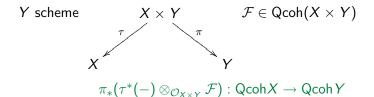
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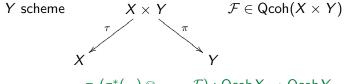
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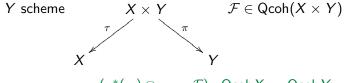
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If  $f: Y \to X$  is a morphism of schemes then

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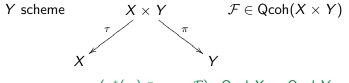
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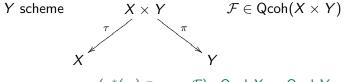
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Let  $X = \mathbb{P}^1$  and  $Y = \operatorname{Spec} k$ . Then  $H^1(X, -) \in \operatorname{Bimod}_k(\operatorname{Qcoh} X, \operatorname{Qcoh} Y)$  is not an integral transform.



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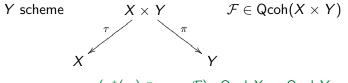
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### Problem

When is  $F \in \text{Bimod}_k(\text{Qcoh}X, \text{Qcoh}Y)$  an integral transform?

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- Qcoh X
- Mod R, R a ring
- Proj A := GrA/TorsA where A is Z-graded

Y, Z non-commutative spaces

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 $Y,\ Z$  non-commutative spaces  $Y\xrightarrow{f} Z$  denotes adjoint pair  $\left(f^*,f_*\right)$  in the diagram

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### Motivation

If  $f: Y \to X$  is a morphism of schemes,  $(f^*, f_*)$  is an adjoint pair.

Adjoint functor theorem  $\Rightarrow$ 

Morphisms  $f: Y \to Z \quad \leftrightarrow \quad \text{Bimod}_k(\text{Mod}Z, \text{Mod}Y).$ 

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and

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are morphisms of noncommutative spaces  $Y \rightarrow X$  and  $X \rightarrow Y$ .

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The latter may not come from a morphism of schemes.

# The Eilenberg-Watts Theorem over Schemes I: The Eilenberg-Watts Functor

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 $v: V \to X$  denotes inclusion of affine open  $Fv_* \in \operatorname{Bimod}_k(\operatorname{Qcoh} V, \operatorname{Qcoh} Y) \Rightarrow$ 

$$\mathsf{Fv}_*\cong -\otimes_{\mathcal{O}_V}\mathcal{F}_V$$

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### Theorem (Van den Bergh, N.)

The collection  $\mathcal{F}_V$  induces (via gluing) a functor

W(-): Bimod<sub>k</sub>(QcohX, QcohY)  $\rightarrow$  QcohX  $\times$  Y

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 If

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$$F \cong \bigoplus_{i=m}^{\infty} \mathsf{H}^1(\mathbb{P}^1, (-)(i))^{\oplus n_i}$$

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Theorem (N)

There is a natural transformation

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What is the structure of obstructions ker  $\Gamma_F$  and cok  $\Gamma_F$ ?

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What is the structure of obstructions ker  $\Gamma_F$  and cok  $\Gamma_F$ ? If  $X = \mathbb{P}^1_k$ ?  $Y = \operatorname{Spec} k$ ? F preserves noetherian objects?

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If  $f: Y \to X$  is morphism of noetherian schemes, then  $f^*$  preserves noetherian objects.

## Abuse of Notation

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$$\Gamma_F: F \to \pi_*(\tau^* - \otimes_{\mathcal{O}_{X \times Y}} W(F)) \equiv H^0(X, - \otimes_{\mathcal{O}_X} W(F)).$$

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Theorem (N)

If  $F \in \mathsf{Bimod}_k(\mathsf{Qcoh}\mathbb{P}^1_k,\mathsf{Mod}k)$  preserves noetherian objects then

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$$F \cong \oplus_{i=-n}^{\infty} H^1(\mathbb{P}^1, (-)(i))^{\oplus n_i} \oplus H^0(\mathbb{P}^1, -\otimes W(F)).$$

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### Corollary

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#### Corollary

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#### Exercise

Use main result to give elementary proof of Serre Duality, i.e.  $\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(-,\mathcal{O})^* \cong H^1(\mathbb{P}^1,-\otimes_{\mathcal{O}_{\mathbb{P}^1}}\mathcal{O}(-2)).$ 

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### Overview of Proof of Main Theorem

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Eilenberg-Watts over Schemes  $\Rightarrow \exists$ 

$$0 \to \ker \Gamma_F \to F \xrightarrow{\Gamma_F} H^0(\mathbb{P}^1, -\otimes W(F)) \to \operatorname{cok} \Gamma_F \to 0$$

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f(n) is eventually constant. Let m = eventual dimension.

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Thus either

• m = 0 in which case

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• m > 0. In this case a contradiction is found.

### The case m > 0

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Prove ker  $\Gamma_F$  is "large" enough to contain non-totally global functor.

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For  $q\in \mathbb{P}^1$ , define  $R_q$  by

$$R_q(-) := H^0(\mathbb{P}^1, ((-)/\mathcal{H}^0_q(-))\otimes k(q)),$$

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- If  $u: U \to \mathbb{P}^1$  is inclusion of an open affine containing q, then  $R_q(u_*\mathcal{O}_U) \neq 0$

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#### Proposition

If 
$$m > 0$$
, there exists  $q \in \mathbb{P}^1$  such that  $R_q \subset \ker \Gamma_F$ .

# Key Observation

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If  $F, G : \operatorname{Qcoh} \mathbb{P}^1 \to \operatorname{Mod} k$  are k-linear, direct limit preserving and G is totally global then  $\Omega : F \to G$  can be constructed inductively.

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commutes  $\forall i \in \mathbb{Z}$  and  $\psi \in \text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{O}(i), \mathcal{O}(i+1))$ . Then  $\exists !$  natural transformation  $\Omega : F \to G$  extending  $\underline{\Omega}$ .

### Question

Can we replace  $\mathbb{P}^1$  by a smooth curve?

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### Thank you for your attention!

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