The Geometry of (Some) Noncommutative Projective Lines

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July 2, 2013

Conventions and Notation

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- L/k finite extension

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- $\overline{L}$ an algebraic closure of L

Part 1

Noncommutative Projective Lines

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- Proj $A := \text{Gr}A/\text{Tors}A$ where A is $\mathbb{Z}$-graded

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Part 2

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Remark

The result holds even if L/k is infinite

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$$V(\lambda) := {}_1L \vee \lambda(L)_\lambda$$

Action defined as $a \cdot v \cdot b := av\lambda(b)$.

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3. What is the relationship between the arithmetic data and the automorphism groups?

Part 3

Noncommutative Symmetric Algebras

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Should have expected left and right Hilbert series

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Problem

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No natural multiplication: if $x, y \in V$, $x \cdot y$ **not** in $\frac{V \otimes V^*}{\mathrm{im}\,\eta_0}$.

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Example

If $(\mathcal{O}(n))_{n\in\mathbb{Z}}$ is seq. of objects in a category A, then

$$A_{ij} = \mathrm{Hom}_{\mathsf{A}}(\mathcal{O}(-j), \mathcal{O}(-i))$$

with mult. $=$ composition makes $\oplus_{i,j\in\mathbb{Z}} A_{ij}$ a $\mathbb{Z}$-algebra

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$$\text{Gr}\mathbb{S}^{n.c.}(V) \equiv \text{Gr}\mathbb{S}(V).$$

Part 4

Arithmetic Noncommutative Projective Lines

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Birational invariants of noncommutative projective lines $\mathbb{P}^{n.c.}(V)$ may suggest birational invariants of a noncommutative surface.

Toy Models

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Thanks Thomas Nevins.

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Part 5

Classification of Noncommutative Projective Lines

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$$V \cong L_\sigma \otimes_L W \otimes_L L_\tau$$

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$\mathbb{P}^{n.c.}(V) \equiv_k \mathbb{P}^{n.c.}(W)$ if and only if there exists $\sigma, \tau \in \mathrm{Gal}(L/k)$ such that either

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($\Leftarrow$) proven in greater generality by I. Mori.

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$\mathcal{O}_{(\sigma_1,\tau_1)} \cap \{(\sigma_2, \tau_2), (\sigma_2^{-1}, \tau_2^{-1}), (\tau_2, \sigma_2), (\tau_2^{-1}, \sigma_2^{-1})\} \neq \emptyset.$

Classification Theorem Version 2, Case 3

Theorem (cont.)

Let $G := \mathrm{Gal}(\overline{L}/L)$. Suppose char $k \neq 2$. Then $\mathbb{P}^{n.c.}(V_1) \equiv \mathbb{P}^{n.c.}(V_2)$ if and only if
Case 3: $\exists\ \lambda_i \in \mathrm{Emb}(L)$ of G-orbit size two, such that

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- $\mathcal{O}_{\lambda_1} \cap \lambda_2^G \neq \emptyset$ or
- $\mathcal{O}_{\lambda_1} \cap \mu_2^G \neq \emptyset$ where $\mu_2 = (\overline{\lambda_2})^{-1}|_L$.

Part 6

Classification of Isomorphisms $\mathbb{P}^{n.c.}(V) \to \mathbb{P}^{n.c.}(W)$

Canonical Equivalences 1

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- $\Phi(M)_i := M_i$ as a set, with $\mathbb{S}^{n.c.}(W)$-module structure

$$\Phi(M)_i \otimes \mathbb{S}^{n.c.}(W)_{ij} \xrightarrow{1\otimes\phi^{-1}} \Phi(M)_i \otimes \mathbb{S}^{n.c.}(V)_{ij} \xrightarrow{\mu} \Phi(M)_j.$$

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Φ descends uniquely to an equivalence $\Phi : \mathbb{P}^{n.c.}(V) \to \mathbb{P}^{n.c.}(W)$.

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$$\zeta_i = \begin{cases} \delta \text{ if } i \text{ is even} \\ \tau \text{ if } i \text{ is odd,} \end{cases}$$

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Notation

$T_{\delta,\tau} : \mathbb{P}^{n.c.}(V) \to \mathbb{P}^{n.c.}(L_{\delta^{-1}} \otimes V \otimes L_\tau)$

Canonical Equivalences 3: Shifts

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Definition of $[i] : \mathrm{Gr}\mathbb{S}^{n.c.}(V) \to \mathrm{Gr}\mathbb{S}^{n.c.}(V)$ $(i \in \mathbb{Z})$:

- $M[i]_j := M_{j+i}$ with multiplication induced from mult. on M
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Remark

$[\Phi]$ also classified.

Part 7

Automorphism Groups

Aut $\mathbb{P}^{n.c.}(V)$, Stab V and Aut V

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The group Aut $\mathbb{P}^{n.c.}(V)$

Aut $\mathbb{P}^{n.c.}(V) :=$ the set equivalence classes of k-linear shift-free equivalences $\mathbb{P}^{n.c.}(V) \to \mathbb{P}^{n.c.}(V)$, with composition induced by composition of functors.

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To describe it: need

Definition of Stab V

Stab V = subgroup of Gal $(L/k) \times$ Gal (L/k) consisting of (σ, τ) such that $L_{\sigma^{-1}} \otimes_L V \otimes_L L_\tau \cong V$

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Aut $V =$ the set of isomorphisms $V \to V$ modulo the relation defined by setting $\phi' \equiv \phi \Leftrightarrow$ there exist $\alpha, \beta \in L^*$ such that $\phi' \circ \phi^{-1}(v) = \alpha \cdot v \cdot \beta$ for all $v \in V$.

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$$\text{Aut } \mathbb{P}^{n.c.}(V) \cong \text{Aut } V \rtimes_{\psi} \text{Stab } V^{op}.$$

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- Aut $V \cong L^* \times L^* / \{(\alpha\sigma(\beta), \alpha\tau(\beta)) | \alpha, \beta \in L^*\}$

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In the special case that V is not simple and Gal(L/k) is cyclic the result was obtained by Kussin.

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Then $\psi : \text{Stab } V \to \text{End (Aut } (V))$ is the homomorphism defined by

$$\psi((g,h))[x] = [\psi_{g,h}(x)].$$

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and derived equivalences are induced by translations and equivalences

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Thank you for your attention!