The Geometry of Noncommutative Curves of Genus Zero

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Part 1

Commutative Curves of Genus Zero

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Question

When is $X \cong \mathbb{P}^1$?

$X = V(f(X,Y,Z))$ **has a k-rational point** if $\exists\ P = [c,d,e] \in \mathbb{P}^2_k$ such that $f(c,d,e) = 0$.

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$$\begin{array}{ccc} X & \xrightarrow{\text{antican. emb.}} & \operatorname{Proj}(\bigoplus_j H^0(X, \omega_X^{*\,\otimes j})) \\ \downarrow & & \uparrow \cong \\ \operatorname{Proj}(\bigoplus_i H^0(X, \mathcal{O}(P)^{\otimes i})) & \xrightarrow[2-Veronese]{} & \operatorname{Proj}(\bigoplus_j H^0(X, \mathcal{O}(P)^{\otimes 2j})) \end{array}$$

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Morally speaking, Tsen's theorem says: over certain base fields, curves of genus zero are projective lines.

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if and only if

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Part 2

Noncommutative Curves of Genus Zero (after Kussin)

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Theorem (Gabriel-Rosenberg)

A (quasi-separated) scheme X can be recovered up to isomorphism from Qcoh$X$.

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Assumption

From now on we will work only with homogeneous H.

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$$\text{geometry of H} \leftrightarrow \text{rep. theory of } \mathrm{End}_{\mathrm{H}}(\mathcal{T}).$$

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Classification of indecomposable bundles in H

Given a line bundle $\mathcal{L}$ over H,

- $\exists!$ indecomposable bundle $\overline{\mathcal{L}}$ s.t. there is an irreducible morphism $\mathcal{L} \to \overline{\mathcal{L}}$
- every indec. bundle is $\cong$ to $\tau^i(\mathcal{L})$ or $\tau^i(\overline{\mathcal{L}})$.

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Question

Which properties of H are dictated by the underlying bimodule?

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In general:

- $\mathrm{End}(\mathcal{L})$ and $\mathrm{End}(\overline{\mathcal{L}})$ will always be division rings f.d. over k.
- The underlying bimodule of H will always have left-right dimensions $(1, 4)$ or $(2, 2)$.

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Theorem (Artin and Zhang (1994))

If $(\mathcal{L}, \sigma)$ is an ample pair, then there is an equivalence $\mathrm{H} \to \operatorname{proj} A := \operatorname{gr} A / \operatorname{tors} A$.

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- Kussin identifies σ such that the homogeneous coordinate ring A is

$$k\langle X, Y, Z\rangle / \langle XY - YX, XZ - ZX, YZ + ZY, Z^2 + aY^2 - cX^2\rangle$$

Part 3

Noncommutative Symmetric Algebras

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Should have expected left and right Hilbert series

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Problem

Too many relations.

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${}^*V := \mathrm{Hom}_L({}_L V, L)$ with action $(a \cdot \phi \cdot b)(x) = b\phi(xa)$.

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Problem

No natural multiplication: if $x, y \in V$, $x \cdot y$ **not** in $\frac{V \otimes V^*}{\operatorname{im} \eta_0}$.

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Example

If $(\mathcal{O}(n))_{n \in \mathbb{Z}}$ is seq. of objects in a category A, then

$$A_{ij} = \mathrm{Hom}_{\mathsf{A}}(\mathcal{O}(j), \mathcal{O}(i))$$

with mult. = composition makes $\oplus_{i,j \in \mathbb{Z}} A_{ij}$ a $\mathbb{Z}$-algebra

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Van den Bergh defines $\mathbb{S}^{nc}(\mathcal{E})$.

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- A is i-**periodic** if $A \cong A(i)$.

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If $R \neq S$, (following Van den Bergh) we let $\mathbb{S}^{nc}(N)_{ii} = R$ if i is even and $\mathbb{S}^{nc}(N)_{ii} = S$ if i is odd.

Part 4

Noncommutative $\mathbb{P}^1$-bundles over Division Rings and Noncommutative Tsen's Theorem

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Proposition (N (2014))

$\mathbb{P}^{nc}(V)$ is a homogeneous noncommutative curve of genus zero.

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Let H be a noncommutative curve of genus zero with underlying bimodule M. Then there is a k-linear equivalence

$$\mathsf{H} \to \mathbb{P}^{nc}(M).$$

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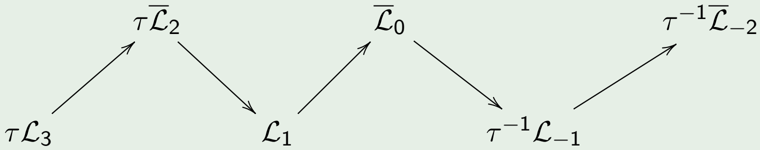

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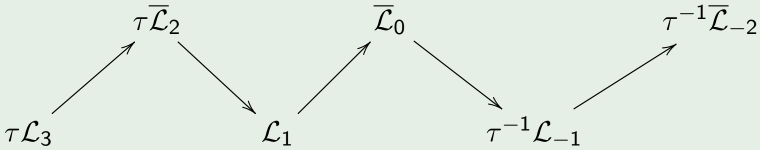

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$$H_{ij} = \begin{cases} \mathrm{Hom}(\mathcal{O}(j), \mathcal{O}(i)) & \text{if } j \geq i \\ 0 & \text{if } i > j \end{cases}$$

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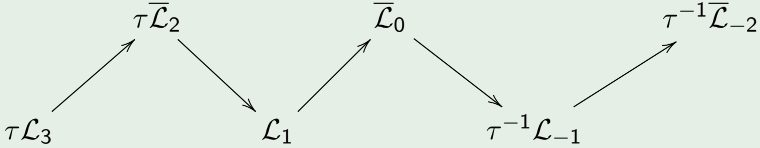

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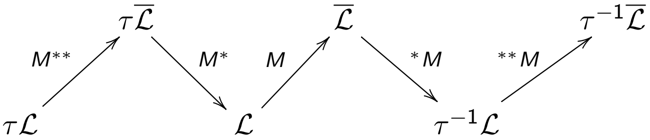
Consequence



Recall the commutative picture when X has rational point P:

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X & \xrightarrow{\text{antican. emb.}} & \mathrm{Proj}(\bigoplus_j H^0(X, {\omega_X^*}^{\otimes j})) \\
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This holds even if $\mathsf{H} = \mathrm{coh}X$ and X doesn't have a rational point.

Part 5

Noncommutative Witt's Theorem

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Corollary (N (2015))

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Thank you for your attention!