Highly Connected Subgraphs of Graphs with Given Independence Number (Extended Abstract)

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Abstract

Let G be a graph on n vertices with independence number α. What is the largest k-connected subgraph that G must contain? We prove that if n is sufficiently large ($n \geq \alpha^2 k + 1$ will do), then G contains a k-connected subgraph on at least n/α vertices. This is sharp, since G might be the disjoint union of α equally-sized cliques. For $k \geq 3$ and $\alpha = 2, 3$, we shall prove that the same result holds for

$n \geq 4(k-1)$ and $n \geq \frac{27(k-1)}{4}$ respectively, and that these lower bounds on n are sharp.

Keywords: Connectivity, highly connected subgraph, independence number

1 Introduction

For any terms not defined here, we refer the reader to [1]. When can we find a large highly connected subgraph of a given graph G? A classical theorem due to Mader [10] (see also [5]) states that if G has average degree at least $4k$, then G contains a k-connected subgraph H. Mader's theorem does not give a lower bound on the order of H. If G is dense (for instance if $\delta(G)$, the minimum degree of G, is bounded below), it is natural to expect that G in fact contains a *large* highly connected subgraph. Using a recent result of Borozan et al. [3], we know that every graph G of order n with $\delta(G) \geq \sqrt{c(k-1)n}$ contains a k-connected subgraph of order at least $\sqrt{(k-1)n/c}$, where $c = 2123/180$. What if we are interested in finding a larger k-connected subgraph, say of order cn? Along these lines, Bollobás and Gyárfás [2] conjectured that any graph G of order $n \geq 4k-3$, or its complement $\overline{G}$, contains a k-connected subgraph H of order at least $n - 2(k-1)$. Since either G or $\overline{G}$ is a dense graph, we might expect to find a very large highly connected subgraph in one of them. This conjecture was settled affirmatively for $n \geq 13k - 15$ by Liu, Morris and Prince [9], and then for $n > 6.5(k-1)$ by Fujita and Magnant [8].

Suppose next that $\delta(G) = cn$. Can we find a k-connected subgraph of G on at least cn vertices? It turns out that the answer is "yes" for sufficiently large n, and in fact a simple argument gives even more. To see this, suppose $n \gg k$ and $n \gg 1/c$, and let $m = \lfloor 1/c \rfloor$. If G itself is not k-connected, then G can be "split" into two pieces with a (negligible) separating set of size at most $k-1$. Both pieces must have order at least cn, so as not to violate the minimum degree condition. Discard one of the pieces, together with the separating set, to obtain a new graph G'. If G' is not k-connected, we continue the process, which terminates after at most $m-1$ steps, leaving a k-connected graph H on at least cn vertices. Now either this graph H, or one of the (at most $m-1$)

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discarded pieces, must have order at least $\frac{n-(k-1)(m-1)}{m} \approx n/m \geq cn$. For instance, if $c > 1/2$, then G itself is k-connected.

Therefore, we instead focus on another graph parameter which forces G to be dense, but which does not immediately yield a trivial bound for our problem. Such a parameter is the independence number $\alpha(G)$. If a graph G has independence number α, then its complement $\overline{G}$ has clique number α, so that, by Turán's theorem, $\overline{G}$ has average degree at most around $(1-1/\alpha)n$, and so G has average degree at least around n/α. It is natural to conjecture that this automatically implies that G has a k-connected subgraph on at least n/α vertices. However, this conjecture is false. Indeed, our graphs in Constructions 1 and 2 (see Section 2) have average degrees $(19/32)n$ and $(307/729)n$, and no k-connected subgraphs of orders $n/2$ and $n/3$ respectively.

Structures in graphs with fixed independence number are widely studied. In particular, the problem of finding a large subgraph with certain properties in a graph with fixed independence number has received much attention. For example, a famous theorem due to Chvátal and Erdős [4] from 1972 states that, any graph G on at least three vertices, whose independence number $\alpha(G)$ is at most its connectivity $\kappa(G)$, contains a Hamiltonian cycle. Motivated by this, Fouquet and Jolivet [6] conjectured in 1976 that if instead, G is a k-connected graph of order n with $\alpha(G) = \alpha \geq k$, then G has a cycle with length at least $\frac{k(n+\alpha-k)}{\alpha}$. Recently, this long standing conjecture was settled affirmatively by O et al. [11].

In this paper, we consider the following question. Fix $k \geq 1$, and let G be a graph on n vertices with independence number α. Can we always find a large k-connected subgraph of G? A little thought shows that, if $n \leq \alpha k$, then there might be no such subgraph, and if $n \geq \alpha k + 1$, then we are only guaranteed a k-connected subgraph of order $\lceil n/\alpha \rceil$, since in both cases G might consist of the disjoint union of α cliques, each with either $\lceil n/\alpha \rceil$ or $\lfloor n/\alpha \rfloor$ vertices. Such a G has the fewest edges among all graphs of independence number α, so it seems that it should be extremal for our problem as well.

In fact, for large n, this construction (which we will call the *disjoint clique construction*, or just DCC) is indeed extremal. Specifically, we prove in Theorem 2.2 that any graph G of order $n \geq \alpha^2 k + 1$ and independence number α must have a k-connected subgraph of order at least $\lceil n/\alpha \rceil$. However, for smaller values of n, this no longer applies. For instance, when $\alpha = 2$ and $k \geq 3$, there is a graph of order $n = 4k - 5$ and independence number 2 with no k-connected subgraph of order at least $\lceil n/2 \rceil$ (see Construction 1). Also, when $\alpha = 3$ and $k \geq 3$, there is a graph of order $n = \lceil \frac{27(k-1)}{4} \rceil - 1$ (for

k odd) or $n = \lceil \frac{27(k-1)}{4} \rceil - 2$ (for k even), and independence number 3, with no k-connected subgraph of order at least $\lceil n/3 \rceil$ (see Construction 2). These examples, however, *are* extremal, in the sense that for any graph with order $n \geq 4k-4$ (resp. $n \geq \lceil \frac{27(k-1)}{4} \rceil$), there is always a k-connected subgraph on at least $\lceil n/2 \rceil$ (resp. $\lceil n/3 \rceil$) vertices, and the DCC is thus optimal. We prove this in Theorem 2.3 and Theorem 2.4. In view of the complexity of both Construction 2 and the proof of Theorem 2.4, we suspect that the generalization to higher values of α is far from simple.

Although our main result (Theorem 2.4) might appear rather modest, the structural difference of Constructions 1 and 2 shows the difficulty of finding the exact lower bound on n for $\alpha \geq 4$. Indeed, we have no conjecture as to what this lower bound might be. In contrast, the sharpness constructions for both the Bollobás-Gyárfás conjecture and the Fouquet-Jolivet conjecture (now the theorem of O et al.) both consist of one essentially unique example, suggesting that our problem is more difficult than these others.

2 Results

In this section, we list our main results. For those who are interested in the proofs of our results, see the full paper in [7] (it is available on request).

From now on we fix $k \geq 1$ and $\alpha \geq 1$. Our first observation is that the case $k = 1$ is trivial. Indeed, if G is a graph of order n and independence number α, then the largest connected component of G must contain at least $\lceil n/\alpha \rceil$ vertices. The case $k = 2$ is a little harder, but is covered by the following:

Proposition 2.1 *Let G be a graph of order $n > 2\alpha$ and independence number α. Then G contains a 2-connected subgraph of order at least $\lceil n/\alpha \rceil$.*

Although this proposition is not difficult to show, due to space limitations, the proof is omitted. The example of a path on 2α vertices shows that the hypothesis cannot be weakened.

Consequently, we may restrict our attention to the case $k \geq 3$. In fact our main interest in this paper is in large values of k and small values of α. Our first main result shows that, for large values of n, the disjoint clique construction (DCC) is optimal.

Theorem 2.2 *Let $k \geq 2, \alpha \geq 2$ and let G be a graph of order $n \geq \alpha^2 k+1$ and independence number α. Then G contains a k-connected subgraph of order at least $\lceil n/\alpha \rceil$.*

When $\alpha = 2$, we can improve the bound in Theorem 2.2 slightly.

Theorem 2.3 *Let $k \geq 3$, and let G be a graph of order $n \geq 4(k-1)$ and independence number 2. Then G has a k-connected subgraph of order at least $\lceil n/2 \rceil$.*

Rather surprisingly, the bound on n in Theorem 2.3 is best possible, as can be seen by the following construction:

Construction 1 *Let $k \geq 3$. Let G be formed from five cliques A, B, C, D and E, of orders $k-1$, $\lfloor \frac{k-1}{2} \rfloor$, $\lceil \frac{k-1}{2} \rceil$, $k-1$ and $k-2$ respectively, and with all edges from A to B, B to D, D to E, E to C and C to A. Then $|G| = 4k-5 = n$, $\alpha(G) = 2$, and the largest k-connected subgraph in G is $G[D \cup E]$, of order $2k-3 < 2k-2 = \lceil n/2 \rceil$.*

For $\alpha = 3$ and $k \geq 3$, we have the following construction:

Construction 2 *Let $k \geq 3$. Let G be formed from nine cliques $A, B, C_1, C_2, D_1, D_2, E_1, E_2$ and F, where*

$$|A| = |D_1| = |D_2| = |E_1| = |E_2| = k-1,$$
$$|B| = \lceil \frac{k+1}{4} \rceil,\ |C_1| = \lfloor \frac{k-1}{2} \rfloor,\ |C_2| = \lfloor \frac{3k-5}{4} \rfloor,\ |F| = \lfloor \frac{k-3}{4} \rfloor,$$

and where we join all vertices in the following 14 pairs of cliques:

$$(A, C_i), (C_1, C_2), (C_i, D_i), (D_i, E_i), (D_i, F), (E_i, F), (B, E_i), (A, B).$$

Then $\alpha(G) = 3$, $|G| = n = \lceil \frac{27(k-1)}{4} \rceil - 1$ if k is odd, and $|G| = n = \lceil \frac{27(k-1)}{4} \rceil - 2$ if k is even. The largest k-connected subgraphs of G are $G[A \cup C_1 \cup C_2]$ and $G[D_i \cup E_i \cup F]$ (for $i = 1, 2$), and it is easy to check that both have at most $\lceil n/3 \rceil - 1$ vertices.

It turns out that for larger values of n the DCC is once again optimal. Specifically, if $k \geq 3$, we can prove the following theorem.

Theorem 2.4 *Let $k \geq 3$, and let G be a graph of order $n \geq \frac{27(k-1)}{4}$ and independence number 3. Then G has a k-connected subgraph of order at least $\lceil n/3 \rceil$.*

3 Conclusion

In the full paper [7] we have shown that, for sufficiently large n, any graph G of order n and independence number α has a k-connected subgraph on at least $\lceil n/\alpha \rceil$ vertices. We have also determined precisely what "sufficiently large" means in the cases $\alpha = 2$ and $\alpha = 3$. In these cases, our lower bounds on n are accompanied by constructions showing that the bounds are best possible. We also presented a construction for larger values of α, which shows that in general we need $n \geq 2\alpha(k-1)$ to guarantee a k-connected subgraph of order at least $\lceil n/\alpha \rceil$. Due to space limitations, this construction is omitted in this extended abstract. The determination of the correct lower bound on n in general remains open.

Finally, here is a related question. Suppose again that n is not in fact large enough to guarantee a k-connected subgraph on at least $\lceil n/\alpha \rceil$ vertices. What is the largest k-connected subgraph (as a function of α, k and n) that G must nonetheless contain?

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